Math 871 Topology I
Worksheet 3

1. Let A, B, and A_α be subsets of a topological space X for each α.

   (a) Show that if $A \subseteq B$, then $\overline{A} \subseteq \overline{B}$.

   (b) Show that
   $$\bigcup_{\alpha\in I} \overline{A_\alpha} \subseteq \overline{\bigcup_{\alpha\in I} A_\alpha}.$$

   (c) Prove or disprove: $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

   (d) Prove or disprove:
   $$\bigcup_{\alpha\in I} \overline{A_\alpha} = \overline{\bigcup_{\alpha\in I} A_\alpha}.$$

2. Let X be an ordered set in the order topology.

   (a) Show that $\overline{(a,b)} \subseteq [a,b]$.

   (b) Give an example of an ordered set X and interval (a,b) such that $\overline{(a,b)} \neq [a,b]$.

   (c) Show that in $\mathbb{R}$ with the standard topology, $\overline{(a,b)} = [a,b]$ for all $a < b$.

3. In the finite complement topology on $\mathbb{R}$, to what points (if any) does the sequence $\left(\frac{1}{n}\right)$ converge?

4. Consider $\mathbb{R}$ with the included point topology with included point 0.

   (a) To what points (if any) does the constant sequence equal to 0 converge to?

   (b) To what points (if any) does the constant sequence equal to 1 converge to?

5. Show that a subspace of a Hausdorff space is Hausdorff.

6. For each of the following topologies on $\mathbb{R}$, determine which ones are T_1 and which ones are Hausdorff:

   (a) $\mathcal{T}_1$ the standard topology.

   (b) $\mathcal{T}_2$ the finite complement topology.

   (c) $\mathcal{T}_3$ the lower limit topology.

   (d) $\mathcal{T}_4 = \{(-\infty, a) \mid a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$ the infinite ray topology.

   (e) $\mathcal{T}_5$ the discrete topology.

   (f) $\mathcal{T}_6$ the indiscrete topology.

7. Show that the line with two origins $L = \mathbb{R} \setminus \{0\} \cup \{\mathbf{0}_1, \mathbf{0}_2\}$ is a T_1 space.

8. Consider following collection of subsets of $\mathbb{R}$:

$$\mathcal{T} = \{U \subseteq \mathbb{R} \mid \mathbb{R} \setminus U \text{ is a countable set } \} \cup \{\emptyset\}.$$

Note that finite sets (including the empty set) are considered to be countable.

(a) Show that $\mathcal{T}$ is a topology on $\mathbb{R}$.

(b) Let (x_n) be a sequence of real numbers. Show that if (x_n) converges in $(R, \mathcal{T})$, then it must be eventually constant.

(c) Show that every convergent sequence in $(\mathbb{R}, \mathcal{T})$ converges to only one limit.

(d) Show that $(\mathbb{R}, \mathcal{T})$ is not Hausdorff.

9. Consider real numbers y and x_n for all $n \geqslant 1$.

(a) In $\mathbb{R}$ in the standard topology, prove that (x_n) converges to y if and only if for all $\varepsilon > 0$ there exists $N \geqslant 1$ such that $|x_n - y| < \varepsilon$ for all $n \geqslant N$.

(b) In $\mathbb{R}_\ell$, prove that (x_n) converges to y if and only if for all $\varepsilon > 0$ there exists N such that $y \leqslant x_n < y + \varepsilon$ for all $n \geqslant N$.

(c) In $\mathbb{R}_{\text{discrete}}$, prove that (x_n) converges to y if and only if there exists N such that $y = x_n$ for all $n \geqslant N$.