Math 871 Topology I
Worksheet 3 solutions

1. Let A, B, and A_α be subsets of a topological space X for each α.

   (a) Show that if $A \subseteq B$, then $\overline{A} \subseteq \overline{B}$.

   Solution. Any closed set C that contains B must contain A, and thus

$$\overline{A} = \bigcap_{\substack{A \subseteq C \\ C \text{ closed}}} C \subseteq \bigcap_{\substack{B \subseteq C \\ C \text{ closed}}} = \overline{B}.$$

   (b) Show that

$$\bigcup_{\alpha \in I} \overline{A_\alpha} \subseteq \overline{\bigcup_{\alpha \in I} A_\alpha}.$$

   Solution. Fix $\beta \in I$. Then by (a),

$$A_\beta \subseteq \bigcup_{\alpha \in I} A_\alpha \implies \overline{A_\beta} \subseteq \overline{\bigcup_{\alpha \in I} A_\alpha}.$$

   Since each $\overline{A_\beta}$ is a subset of the right hand side, so is their union. Thus

$$\bigcup_{\alpha \in I} \overline{A_\alpha} \subseteq \overline{\bigcup_{\alpha \in I} A_\alpha}.$$

   (c) Prove or disprove: $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

   Solution. True. The containment ($\supseteq$) is a special case of (b).
   ($\subseteq$) By (a), $A \subseteq \overline{A}$ and $B \subseteq \overline{B}$, so

$$A \cup B \subseteq \overline{A} \cup \overline{B}.$$

   The finite union of closed sets is closed, so $\overline{A} \cup \overline{B}$ is closed. Any closed set that contains $A \cup B$ also contains its closure, and thus

$$\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}.$$

   (d) Prove or disprove:

$$\bigcup_{\alpha \in I} \overline{A_\alpha} = \overline{\bigcup_{\alpha \in I} A_\alpha}.$$

   Solution. This is false. Consider $\mathbb{R}$ with the standard topology, and the sets

$$A_n = \left(\frac{1}{n}, 1\right]$$

   with n ranging over all the positive integers. For each n,

$$\overline{A_n} = \overline{\left(\frac{1}{n}, 1\right]} = \left[\frac{1}{n}, 1\right].$$

Thus
$$\bigcup_{n\geqslant 1}\overline{A_n} = \bigcup_{n\geqslant 1}\left[\frac{1}{n},1\right] = (0,1].$$
On the other hand,
$$\bigcup_{n\geqslant 1} A_n = \bigcup_{n\geqslant 1}\left(\frac{1}{n},1\right] = (0,1],$$
and
$$\overline{\bigcup_{n\geqslant 1} A_n} = \overline{(0,1]} = [0,1].$$

2. Let X be an ordered set in the order topology.

   (a) Show that $\overline{(a,b)} \subseteq [a,b]$.

   Solution. Suppose that $x < a$. If x is the smallest element in X, then $[x,a)$ is an open neighborhood of x and $[x,a)\cap(a,b)=\emptyset$. If not, then there exists $y<x$ in X, and $x\in(y,a)$ is an open set with $(y,a)\cap(a,b)=\emptyset$. Thus $x\notin\overline{(a,b)}$. Similarly, we can show that any $x>b$ is not in $[a,b]$. Therefore, $\overline{(a,b)}\subseteq[a,b]$.

   (b) Give an example of an ordered set X and interval (a,b) such that $\overline{(a,b)}\neq[a,b]$.

   Solution. Consider $\mathbb{Z}$ with the usual order. Then $(0,1)=\emptyset$ is closed, so
   $$\overline{(0,1)}=\emptyset.$$
   On the other hand, $0\in[0,1]$, and thus $[0,1]\neq\emptyset=\overline{(0,1)}$.
   If this feels like cheating, here is an example where (a,b) is nonempty. In the same ordered set, the order topology coincides with the discrete topology: for any $n\in\mathbb{Z}$, $(n-1,n+1)=\{n\}$ is open. Thus $\{n\}$ is also closed, and thus $\overline{(n-1,n+1)}=(n-1,n+1)$.

   (c) Show that in $\mathbb{R}$ with the standard topology, $\overline{(a,b)}=[a,b]$ for all $a<b$.

   Solution. For all $\varepsilon>0$, $(a-\varepsilon,a+\varepsilon)\cap(a,b)\neq\emptyset$, so $a\in\overline{a,b}$. Similarly, for all $\varepsilon>0$, $(b-\varepsilon,b+\varepsilon)\cap(a,b)\neq\emptyset$, so $b\in\overline{(a,b)}$. Since any set is contained in its closure, we conclude that $[a,b]\subseteq\overline{(a,b)}$. By (a), the reverse containment holds, so $[a,b]=\overline{(a,b)}$.

3. In the finite complement topology on $\mathbb{R}$, to what points (if any) does the sequence $\left(\frac{1}{n}\right)$ converge?

Solution. We claim that this sequence converges to every point in $\mathbb{R}$. Indeed, consider any real number x, and any open neighborhood U of x. Then U contains all but finitely many real numbers, and thus in particular since the set
$$K=\left\{\frac{1}{n}\mid n\geqslant 1\right\}$$
is infinite, U must contain all but finitely many elements of K. In particular, there exists some N such that $\frac{1}{n}\in U$ for all $n\geqslant N$. Thus $\left(\frac{1}{n}\right)$ converges to x.

4. Consider $\mathbb{R}$ with the included point topology with included point 0.

   (a) To what points (if any) does the constant sequence equal to 0 converge to?

   Solution. By definition, any nonempty open set includes 0. Therefore, given any point x, and any open neighborhood U of x, since $0 \in U$ then the constant sequence equal to 0 converges to x.

   (b) To what points (if any) does the constant sequence equal to 1 converge to?

   Solution. The constant sequence equal to 1 converges to 1, as any open neighborhood of 1 contains all the values of the sequence. On the other hand, given any point $x \neq 1$, the set $\{x, 0\}$ is open, and does not contain any of the values of the sequence, so the sequence does not converge to x.

5. Show that a subspace of a Hausdorff space is Hausdorff.

   Solution. Let X be a Hausdorff space and consider $A \subseteq X$ under the order topology. Given two distinct points $a, b \in A$, since X is Hausdorff there exists open sets U and V in X such that $a \in U$, $b \in V$, and $U \cap V = \emptyset$. By definition of the subspae topology, $U \cap A$ and $V \cap A$ are open in A. Moreover, $a \in U \cap A$ and $b \in V \cap B$, and

$$(U \cap A) \cap (V \cap A) \subseteq U \cap V = \emptyset$$

   so $U \cap A$ and $V \cap A$ are disjoint. We conclude that A is Hausdorff.

6. For each of the following topologies on $\mathbb{R}$, determine which ones are T_1 and which ones are Hausdorff:

   (a) $\mathcal{T}_1$ the standard topology.

   (b) $\mathcal{T}_2$ the finite complement topology.

   (c) $\mathcal{T}_3$ the lower limit topology.

   (d) $\mathcal{T}_4 = \{(-\infty, a) \mid a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$ the infinite ray topology.

   (e) $\mathcal{T}_5$ the discrete topology.

   (f) $\mathcal{T}_6$ the indiscrete topology.

   Solution.

   (a) $(\mathbb{R}, \mathcal{T}_1)$ is Hausdorff and thus T_1: given $x \neq y$, say $x < y$, we can take

$$\varepsilon := \frac{|x-y|}{2},$$

   which is positive, and

$$U = (x-\varepsilon, x+\varepsilon) \qquad \text{and} \qquad V = (y-\varepsilon, y+\varepsilon)$$

   are disjoint open neighborhoods of x and y respectively.

(b) $(\mathbb{R}, \mathcal{T}_2)$ is T_1 but not Hausdorff: given any two nonempty open sets U and V, the complement of their intersection

$$\mathbb{R} \setminus (U \cap V) = (\mathbb{R} \setminus U) \cup (\mathbb{R} \setminus V)$$

is a union of two finite sets, and thus finite. In particular, $U \cap V$ is necessarily infinite, and thus nonempty. This shows that this topology cannot be Hausdorff. On the other hand, all finite sets are closed by definition, and thus this topology is $\mathcal{T}_1$.

(c) $(\mathbb{R}, \mathcal{T}_3)$ is Hausdorff and thus T_1: given $x \neq y$, say $x < y$, we can take any positive real number ε with

$$\varepsilon < |x - y|,$$

and

$$U = [x, x + \varepsilon) \qquad \text{and} \qquad V = [y, y + \varepsilon)$$

are disjoint open neighborhoods of x and y respectively.

(d) $(\mathbb{R}, \mathcal{T}_4)$ is not T_1, and thus not Hausdorff: for example, $\{0\}$ is not closed since any open neighborhood of 1 contains $U = (-\infty, b)$ for some $b > 1$, and $0 \in U$. In particular, $1 \in \overline{\{0\}}$.

(e) $(\mathbb{R}, \mathcal{T}_5)$ is Hausdorff and thus T_1: given any $x \neq y$, we can take the open sets $U = \{x\}$ and $V = \{y\}$ and $x \in U$, $y \in V$, and $U \cap V = \emptyset$.

(f) $(\mathbb{R}, \mathcal{T}_6)$ is not $\mathcal{T}_1$ and thus not Hausdorff: given any two distinct points, say $x = 0$ and $y = 1$, the only open neighborhood of x is $\mathbb{R}$, which contains y.

7. Show that the line with two origins $L = \mathbb{R} \setminus \{0\} \cup \{\mathbf{0}_1, \mathbf{0}_2\}$ is a T_1 space.

Solution. Set $\mathbb{R}_1 = L \setminus \{\mathbf{0}_2\}$ and $\mathbb{R}_2 = L \setminus \{\mathbf{0}_1\}$. Recall that $U \subseteq L$ is open in L if $U \cap \mathbb{R}_1$ is open in $\mathbb{R}_1$ with the standard topology of $\mathbb{R}$ and $U \cap \mathbb{R}_2$ is open in $\mathbb{R}_2$ with the standard topology of $\mathbb{R}$.

Let a be a nonzero real number. Identifying $\mathbb{R}_1$ with the usual real line,

$$(L \setminus \{a\}) \cap \mathbb{R}_i = \mathbb{R} \setminus \{a\} \quad \text{for both } i = 1, 2.$$

In $\mathbb{R}$ with the standard topology, $\{a\}$ is closed because

$$\mathbb{R} \setminus \{a\} = (-\infty, a) \cup (a, \infty).$$

Thus $\{a\}$ is closed in L.

Let $a = \mathbf{0}_1$. We have

$$(L \setminus \{a\}) \cap \mathbb{R}_1 = \mathbb{R}_1 \setminus \{0\} = (-\infty, 0) \cup (0, \infty),$$

which is open in $\mathbb{R}_1$, and

$$(L \setminus \{a\}) \cap \mathbb{R}_2 = \mathbb{R}_2,$$

which is open in $\mathbb{R}_2$.

8. Consider following collection of subsets of $\mathbb{R}$:

$$\mathcal{T} = \{U \subseteq \mathbb{R} \mid \mathbb{R} \setminus U \text{ is a countable set } \} \cup \{\emptyset\}.$$

Note that finite sets (including the empty set) are considered to be countable.

(a) Show that $\mathcal{T}$ is a topology on $\mathbb{R}$.

Solution. By definition, $\emptyset \in \mathcal{T}$, and since $\mathbb{R} \setminus \mathbb{R} = \emptyset$ is a countable set, $\mathbb{R} \in \mathcal{T}$.
Let $U, V \in \mathcal{T}$. If one of them is empty, then the intersection is empty, and thus in $\mathcal{T}$, so assume they are both nonempty. Then

$$\mathbb{R} \setminus (U \cap V) = (\mathbb{R} \setminus U) \cup (\mathbb{R} \setminus V)$$

is the union of two countable sets, and thus countable. We conclude that $U \cap V \in \mathcal{T}$.
Finally, let $\{U_\alpha\}_{\alpha \in I}$ be a collection of sets in $\mathcal{T}$. If one of them is empty, then it has no effect on the union, so we might as well assume they are all nonempty. Then for any fixed $\beta \in I$,

$$\mathbb{R} \setminus \bigcap_{\alpha \in I} U_\alpha = \bigcup_{\alpha \in I} \mathbb{R} \setminus U_\alpha \subseteq \mathbb{R} \setminus U_\beta$$

is an intersection of countable sets, and thus countable – for example, because it is contained in the countable set $\mathbb{R} \setminus U_\alpha$.

(b) Let (x_n) be a sequence of real numbers. Show that if (x_n) converges in $(R, \mathcal{T})$, then it must be eventually constant.

Solution. Suppose that (x_n) converges to x. The set $C := \{x_n \neq x \mid n \geqslant 1\}$ is countable, and thus closed. Thus $U = X \setminus C$ is an open neighborhood of x. Since (x_n) converges to x, there exists N such that for all $n \geqslant N$, $x_n \in U$. But $x_n \in U \iff x_n = x$. Therefore, $x_n = x$ for all $n \geqslant N$, and thus (x_n) is eventually constant.

(c) Show that every convergent sequence in $(\mathbb{R}, \mathcal{T})$ converges to only one limit.

Solution. Consider any sequence (x_n) that converges. By (b), (x_n) is eventually constant, say with $x_n = x$ for all $n \geqslant N$. Given any $y \neq x$, $U = \mathbb{R} \setminus \{x\}$ is an open neighborhood of y, and $x_n \notin U$ for all $n \geqslant N$. In particular, (x_n) does not have any limit besides y.

(d) Show that $(\mathbb{R}, \mathcal{T})$ is not Hausdorff.

Solution. Let U and V be any two nonemopty open sets. If $U \cap V = \emptyset$, then

$$(X \setminus U) \cup (X \setminus V) = X \setminus (U \cap V) = X.$$

But $X \setminus U$ and $X \setminus V$ are both countable sets, and the union of two countable sets is countable.

9. Consider real numbers y and x_n for all $n \geqslant 1$.

   (a) In $\mathbb{R}$ in the standard topology, prove that (x_n) converges to y if and only if for all $\varepsilon > 0$ there exists $N \geqslant 1$ such that $|x_n - y| < \varepsilon$ for all $n \geqslant N$.

   Solution. The condition that (x_n) converges to y is equivalent to asking that for all basic open neighborhoods U of y, there exists N such that $x_n \in U$ for all $n \geqslant N$. Such U are of them for $U = (y - \varepsilon, y + \varepsilon)$ for $\varepsilon > 0$, so equivalently for all ε there exists N such that

   $$x_n \in (y-\varepsilon, y+\varepsilon) \implies y-\varepsilon < x_n < y+\varepsilon \iff -\varepsilon < x_n - y < \varepsilon \iff |x_n - y| < \varepsilon.$$

   (b) In $\mathbb{R}_\ell$, prove that (x_n) converges to y if and only if for all $\varepsilon > 0$ there exists N such that $y \leqslant x_n < y + \varepsilon$ for all $n \geqslant N$.

   Solution. Suppose that (x_n) converges to y, and fix $\varepsilon > 0$. Since $[y, y + \varepsilon)$ is an open neighborhood of y, we conclude that there exists N such that for all $n \geqslant N$,

   $$x_n \in [y, y + \varepsilon) \iff y \leqslant x_n < y + \varepsilon.$$

   Conversely, if for all ε there exists N such that $y \leqslant x_n < \varepsilon$ for all $n \geqslant N$, then given any open neighborhood U of y, U contains $[z, w)$ for some $z \leqslant y < w$, and so taking $\varepsilon = |w - y|$, we obtain N such that for all $n \geqslant N$

   $$y \leqslant x_n < y + \varepsilon \implies z \leqslant x_n < w \implies x_n \in U.$$

   (c) In $\mathbb{R}_{\text{discrete}}$, prove that (x_n) converges to y if and only if there exists N such that $y = x_n$ for all $n \geqslant N$.

   Solution. Suppose (x_n) converges to y. Then in particular, taking the neighborhood $U = \{y\}$ of y, there exists N such that $x_n \in U$ for all $n \geqslant N$, and thus $x_n = y$ for all such n.

   Conversely, if there exists N such that $x_n = y$ for all $n \geqslant N$, then given any open neighborhood U of y, then for all $n \geqslant N$ we have

   $$x_n = y \in U.$$