

Math 871 Topology I

Worksheet 6

1. Let X be a topological space such that for all distinct x and y in X , there exists a Hausdorff space Y (depending on x and y) and a continuous function $f: X \rightarrow Y$ such that $f(x) \neq f(y)$. Show that X is Hausdorff.
2. Let $f: X \rightarrow Y$ be a continuous function. Show that if $f(A)$ is closed for some $A \subseteq X$, then $f(\overline{A}) = f(A)$.
3. Given a function $f: X \rightarrow Y$ between topological spaces, the **graph** of f is the set

$$\Gamma_f = \{(x, f(x)) \mid x \in X\} \subseteq X \times Y.$$

- (a) Show that if f is continuous and Y is Hausdorff, then Γ_f is closed.
 - (b) Give an example of a continuous function that does not have a closed graph.
4. Show that if $f: X \rightarrow Y$ is a continuous function and $B \subseteq Y$, then

$$\text{Bd}(f^{-1}(B)) \subseteq f^{-1}(\text{Bd}(B)).$$

5. Let $\mathbb{R}^\infty$ be the subset of $\mathbb{R}^\omega$ consisting of all sequences that are eventually zero, that is, all sequences $(x_n)_{n \geq 1}$ such that $x_n \neq 0$ for only finitely many values of n .
 - (a) What is the closure of $\mathbb{R}^\infty$ in $\mathbb{R}^\omega$ equipped with the product topology?
 - (b) What is the closure of $\mathbb{R}^\infty$ in $\mathbb{R}^\omega$ equipped with the box topology?
6. Endow $\mathbb{R}$ with the standard topology and let I be any infinite set. Show that the product and uniform topologies on $\mathbb{R}^I$ do not coincide.