

Math 871 Topology I Worksheet 6 solutions

1. Let X be a topological space such that for all distinct x and y in X , there exists a Hausdorff space Y (depending on x and y) and a continuous function $f: X \rightarrow Y$ such that $f(x) \neq f(y)$. Show that X is Hausdorff.

Solution. Consider any two distinct points x and y in X . By assumption, we can find a Hausdorff space Y and a continuous function $f: X \rightarrow Y$ such that $f(x) \neq f(y)$. Since Y is Hausdorff, there exist open sets U and V in Y such that

$$x \in U, \quad y \in V, \quad U \cap V = \emptyset.$$

Since f is continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are open. Moreover, $x \in f^{-1}(U)$ and $y \in f^{-1}(V)$. Finally, if $z \in U \cap V$ then $f(z) \in U \cap V$, and thus $f^{-1}(U) \cap f^{-1}(V) = \emptyset$. We conclude that X is Hausdorff.

2. Let $f: X \rightarrow Y$ be a continuous function. Show that if $f(A)$ is closed for some $A \subseteq X$, then $f(\overline{A}) = \overline{f(A)}$.

Solution. ($\subseteq$) We showed in class that if f is continuous,

$$f(\overline{A}) \subseteq \overline{f(A)}$$

for all $A \subseteq X$. Here is the proof: given $x \in \overline{A}$, every open neighborhood of x in X intersects A nontrivially. Let V be an open neighborhood of $f(x)$. Since f is continuous, $f^{-1}(V)$ is an open neighborhood of x in X . Since $x \in \overline{A}$, $f^{-1}(V) \cap A$ is nonempty. Therefore, $V \cap f(A)$ is nonempty, and thus $f(x) \in \overline{f(A)}$. This shows that $f(\overline{A}) \subseteq \overline{f(A)}$.

If $f(A)$ is closed, then $\overline{f(A)} = f(A)$.

($\supseteq$) On the other hand, $A \subseteq \overline{A}$ and thus $f(A) \subseteq f(\overline{A})$.

3. Given a function $f: X \rightarrow Y$ between topological spaces, the graph of f is the set

$$\Gamma_f = \{(x, f(x)) \mid x \in X\} \subseteq X \times Y.$$

- (a) Show that if f is continuous and Y is Hausdorff, then Γ_f is closed.

Solution. Let $(x, y) \notin \Gamma_f$, so that $y \neq f(x)$. Since Y is Hausdorff, there exist open sets $V \ni y$ and $U \ni f(x)$ such that $U \cap V = \emptyset$. Since $f(x) \in U$, then $x \in f^{-1}(U)$. Since f is continuous and U is open, then $f^{-1}(U)$ is open, so $f^{-1}(U)$ is an open neighborhood of x .

Let $W = f^{-1}(U) \times V$, which is an open neighborhood of (x, y) . We claim that $W \cap \Gamma_f = \emptyset$. Indeed, if $(a, b) \in W$, then $f(a) \in U$ and $b \in V$, and since $U \cap V = \emptyset$, then in particular $f(a) \neq b$, so $(a, b) \notin \Gamma_f$. Therefore, W is an open neighborhood of (x, y) that does not intersect Γ_f . Since this holds for all $(x, y) \notin \Gamma_f$, we conclude that Γ_f must then be closed.

- (b) Give an example of a continuous function that does not have a closed graph.

Solution. Let X be a non-Hausdorff space; for example, $X = \{a, b\}$ with the trivial topology. The identity function $\text{id}: X \rightarrow X$ is continuous, as both spaces have the same topology. However, the graph of X is

$$\Delta = \{(x, \text{id}(x)) \mid x \in X\} = \{(x, x) \mid x \in X\}$$

and we showed in a problem set that Δ is not closed whenever X is not Hausdorff.

4. Show that if $f: X \rightarrow Y$ is a continuous function and $B \subseteq Y$, then

$$\text{Bd}(f^{-1}(B)) \subseteq f^{-1}(\text{Bd}(B)).$$

Solution. Let

$$x \in \text{Bd}(f^{-1}(B)) = \overline{f^{-1}(B)} \cap X \setminus \overline{f^{-1}(B)}.$$

We need to show that

$$f(x) \in \text{Bd}(B) = \overline{B} \cap \overline{Y \setminus B}.$$

Let U be an open neighborhood of $f(x)$ in Y . Since f is continuous, $f^{-1}(U)$ is an open neighborhood of x in X . Since $x \in \overline{f^{-1}(B)}$, then $f^{-1}(U) \cap f^{-1}(B) \neq \emptyset$. In particular, we can find $z \in f^{-1}(U) \cap f^{-1}(B)$, and

$$f(z) \in U \cap B \implies U \cap B \neq \emptyset.$$

This shows that $f(x) \in \overline{B}$.

5. Let $\mathbb{R}^\infty$ be the subset of $\mathbb{R}^\omega$ consisting of all sequences that are eventually zero, that is, all sequences $(x_n)_{n \geq 1}$ such that $x_i \neq 0$ for only finitely many values of i .

- (a) What is the closure of $\mathbb{R}^\infty$ in $\mathbb{R}^\omega$ equipped with the product topology?

Solution. We claim that $\overline{\mathbb{R}^\infty} = \mathbb{R}^\omega$ in the product topology. Fix $x \in \mathbb{R}^\omega$, and let B be a basic open set, of the form

$$B = \prod_{i=1}^{\infty} B_i$$

where $B_i = \mathbb{R}$ for all but finitely many indices, say $i_1, \dots, i_n$. We will show that if $x \in B$, then B intersects $\mathbb{R}^\infty$ nontrivially, and thus $x \in \overline{\mathbb{R}^\infty}$.

So suppose that $x \in B$. Let $N = \max\{i_1, \dots, i_n\}$, and let $y \in \mathbb{R}^\infty$ be defined by

$$y_i = \begin{cases} x_i & \text{if } i \leq N \\ 0 & \text{otherwise.} \end{cases}$$

By construction, we have that $y_i \in B_i$ for all $i \leq N$. Moreover, $B_i = \mathbb{R}$ for all $i > N$, so that $y_i = 0 \in B_i$ for all $i > N$ as well. It follows that $y \in B$, implying that $x \in \overline{\mathbb{R}^\infty}$. Since $x \in \mathbb{R}^\omega$ was arbitrary, we conclude that $\overline{\mathbb{R}^\infty} = \mathbb{R}^\omega$.

(b) What is the closure of $\mathbb{R}^\infty$ in $\mathbb{R}^\omega$ equipped with the box topology?

Solution. We claim that $\mathbb{R}^\infty$ is closed in $\mathbb{R}^\omega$ with the box topology, so that $\overline{\mathbb{R}^\infty} = \mathbb{R}^\infty$. Let $x \in \mathbb{R}^\omega$ such that $x \notin \mathbb{R}^\infty$. Then there are infinitely many indices, say $i_1, i_2, i_3, \dots$, such that $x_{i_n} \neq 0$ for all $n \geq 1$. Let

$$U = \prod_{i=1}^{\infty} U_i \quad \text{with} \quad U_i = \begin{cases} \mathbb{R}^* = \mathbb{R} - \{0\} & \text{if } i = i_n \text{ for some } n \geq 1 \\ \mathbb{R} & \text{otherwise} \end{cases}.$$

Then U_i is open in $\mathbb{R}$ for all indices i , so that U is open in the box topology. Additionally, $x_{i_n} \neq 0$ for all n implies that $x_i \in U_i$ for all i , so that $x \in U$. Finally, we note that for any $y \in U$, we have that $y_{i_n} \neq 0$ for all $n \geq 1$, and thus y is not eventually zero, and thus $y \notin \mathbb{R}^\infty$.

We conclude that U is a neighborhood of x such that $U \cap \mathbb{R}^\infty = \emptyset$, so that $x \notin \overline{\mathbb{R}^\infty}$. Since our choice of x was arbitrary, it follows that $\mathbb{R}^\infty = \overline{\mathbb{R}^\infty}$, completing the proof.

6. Endow $\mathbb{R}$ with the standard topology and let I be any infinite set. Show that the product and uniform topologies on $\mathbb{R}^I$ do not coincide.

Solution. In $\mathbb{R}^I$ with the uniform metric $\bar{\rho}$, consider the point $z = (0)$, and the open ball $B_{\frac{1}{2}}(z)$. Any basic open neighborhood of z in the product topology, of the form

$$U = \prod_{\alpha \in I} U_\alpha$$

in the product topology, with $U_\alpha = (a_\alpha, b_\alpha)$ for finitely many α , say $\alpha \in \{\alpha_1, \dots, \alpha_N\}$, and $U_\alpha = \mathbb{R}$ for all other α , contains the point

$$y = (y_\alpha)_{\alpha \in I} \quad \text{with} \quad y_\alpha = \begin{cases} 0 & \alpha \in \{\alpha_1, \dots, \alpha_N\} \\ 1 & \alpha \notin \{\alpha_1, \dots, \alpha_N\}. \end{cases}$$

However,

$$\bar{\rho}(z, y) = 1 \implies y \notin B_{\frac{1}{2}}(z).$$

This shows that the product and uniform topologies on $\mathbb{R}^I$ are distinct.