Math 871 Topology I
Our favorite topologies

Definition 1. *Let X be a set. A* **topology** *on X is a collection $\mathcal{T}$ of subsets of X, called* **open sets***, satisfying the following properties:*

- *The empty set belongs to $\mathcal{T}$: $\emptyset \in \mathcal{T}$.*
- *The entire space X belongs to $\mathcal{T}$: $X \in \mathcal{T}$.*
- *If U and V are in $\mathcal{T}$, then so is $U \cap V$.*
- *If $\{U_\alpha\}_{\alpha\in I}$ is any collection of open sets in $\mathcal{T}$, then $\bigcup_{\alpha\in I} U_\alpha \in \mathcal{T}$.*

A **topological space** *is a pair $(X, \mathcal{T})$ where $\mathcal{T}$ is a topology on the set $\mathcal{T}$.*

Here is a list of some of our favorite topologies on any set X:

- The **discrete topology** on X is the topology $\mathcal{T} = \mathcal{P}(\mathcal{T})$ consisting of all the subsets of X.

- The **trivial topology**, also known as the **indiscrete topology** on X, is the topology $\mathcal{T} = \{\emptyset, X\}$ consisting of only the empty set and X.

- The **finite complement topology** on X is the topology

$$\mathcal{T} = \{U \subseteq X \mid X \setminus U \text{ is finite}\} \cup \{\emptyset\}$$

  where the open sets are the empty set and all the sets that contain all but a finite number of elements of X.

- The **included point topology** on X with included point a is the topology given by

$$\mathcal{T} = \{U \subseteq X \mid a \in U\} \cup \{\emptyset\}.$$

- The **excluded point topology** on X with excluded point a is the topology

$$\mathcal{T} = \{U \subseteq X \mid a \notin U\} \cup \{X\}.$$

Here are some of our favorite topologies on $\mathbb{R}$:

- In the **Euclidean topology** or **standard topology** on $\mathbb{R}$, a subset U of $\mathbb{R}$ is *open* if for all $x \in U$ there is a positive real number $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \subseteq U$.

- The **infinite ray topology** on $\mathbb{R}$ is given by

$$\mathcal{T} = \{(-\infty, a) \mid a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}.$$

- The **lower limit topology** on $\mathbb{R}$ has basis

  $$\{[a,b) \mid a<b\}.$$

  We denote the topological space given by $\mathbb{R}$ with the lower limit topology by $\mathbb{R}_\ell$.

- The **K-topology** on $\mathbb{R}$ has basis

  $$\{(a,b) \mid a<b\} \cup \{(a,b) \setminus K \mid a<b\}$$

  where

  $$K := \{\frac{1}{n} \mid n \in \mathbb{Z}, n \geqslant 1\}.$$

  We denote the topological space given by $\mathbb{R}$ with the K-topology by $\mathbb{R}_\ell$.

Other fun topological spaces:

- Let X be the real line with two rival origins; more precisely, we add in a point to the real line, and call our two origins 0_1 and 0_2:

  $$X = (\mathbb{R} \setminus \{0\}) \cup \{0_1, 0_2\}.$$

  If one removes 0_1 from X, we get the set $(\mathbb{R} \setminus \{0\}) \cup \{0_2\}$, which will identify with $\mathbb{R}$ itself in the natural way, by identifying 0_2 with 0, and similarly we may identify $X \setminus \{0_2\}$ with $\mathbb{R}$. The **line with two origins** is the topological space where a subset U of X is open if $U \setminus \{0_1\}$ is an open subset of $X \setminus \{0_1\}$ equipped with the standard topology and $U \setminus \{0_2\}$ is an open subset of $X \setminus \{0_2\}$ equipped with the standard topology.