Problem Set 1

Instructions: For full credit, turn in 5 problems, split between a pdf and a .m2 file (one of each). You are welcome to work together with your classmates on all the problems, and I will be happy to give you hints or discuss the problems with you, but you should write up your solutions by yourself. You cannot use any resources besides me, your classmates, our course notes, and the Macaulay2 documentation.

Problem 1. Install Macaulay2.

Problem 2 (Subalgebras)**.** In Macaulay2, set up the following rings:

a) The $\mathbb{Q}$-algebra $\mathbb{Q}[xy, xu, yv, uv] \subseteq \mathbb{Q}[x, y, u, v]$.

b) The k-algebra U, where $k = \mathbb{F}_{73}$ and

$$U = k \begin{bmatrix} ux & uy & uz \\ vx & vy & vz \end{bmatrix} \subseteq \frac{k[u, v, x, y, z]}{(x^3 + y^3 + z^3)}.$$

Problem 3 (Modules)**.** Consider the domain $R = \mathbb{Q}[x, y, z, a, b, c]/(xb - ac, yc - bz, xc - az)$. Set up the following R-modules, making sure Macaulay2 actually sees them as modules over R:

a) The ideal $I = (x, y, z)$ viewed as an R-module.

b) The R-module $N = \mathbb{Q}$.

c) The 2-generated R-module $M = Rf + Rg$, where the generators f, g satisfy the relations

$$af - xg = 0 \quad bf - yg = 0 \quad cf - zg = 0.$$

d) The submodule of R^3 generated by (a, b, c) and (x, y, z).

Problem 4. Fix a field k.

a) Fix an integer $n > 0$. Show that x^{2n-1}, $x^{2n} + x$, and x^{2n+1} generate $k[x]$ as a k-algebra.

In contrast, we will show that $x^2, x^3 + x$ do not generated $k[x]$ as a k-algebra.

b) Consider the k-algebras $A := k[x^2] \subseteq B := k[x^2, x^3 + x] \subseteq k[x]$. Show that every element in B can be written uniquely as $f + g(x^3 + x)$ for some $f, g \in A$.

c) Show that $x^2, x^3 + x$ do not generated $k[x]$ as a k-algebra.

Problem 5. Find a generating set for $\mathbb{R}[x, y, z]$ as a module over $\mathbb{R}[x^2 + yz, x + y, z]$ of the smallest possible size.

Problem 6. Let k be a field. Find the integral closure of $k[x^{13}, y^{17} + y^7, z^{23} + z^{11}]$ in $k[x, y, z]$.

Problem 7. Show that if R is a unique factorization domain, then R is normal.