Commutative Algebra II

Math 906 Spring 2026

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Warning!

Proceed with caution. These notes are under construction and are 100% guaranteed to contain typos. If you find any typos or errors, I will be most grateful to you for letting me know.

A note on external references

These notes make many direct references to my Commutative Algebra I and Homological Algebra notes, the content of which is assumed to be known to the reader.

Acknowledgements

These notes take much inspiration from other sources, including Tom Marley's notes from Spring 2024 Math 906, from the classic Commutative Algebra books [Mat89, Mat80, Eis95, AM69], and especially from Bruns and Herzog's book *Cohen-Macaulay rings* [BH93]. The chapter on completions very closely follows Tom's notes and the completion section of Jack Jeffries' notes on Kähler differentials.

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Contents

Chapter 0

Setup

Throughout, all rings are commutative with identity 1, all modules are unital, and all ring homomorphisms send 1 to 1. Moreover, all rings will be assumed to be noetherian.

When k is a field, the polynomial ring $R = k[x_1, \dots, x_n]$ can be given an $\mathbb{N}$-grading by setting $\deg(x_i) = d_i$ for some $d_i \in \mathbb{N}$. The most common $\mathbb{N}$-grading, also known as the **standard grading**, is the one where we declare $\deg(x_i) = 1$ for all i. Once we declare the degrees of the variables, we can extend that grading to all monomials as follows:

$$\deg\left(x_1^{a_1} \cdots x_n^{a_n}\right) = a_1 d_1 + \cdots + a_n d_n.$$

In this case, a **homogeneous element** in R is any k-linear combination of monomials of the same degree. We write R_i for the set of all homogeneous elements of degree i, which is an abelian group under addition, and note that

$$R = \bigoplus_i R_i.$$

Note also that $R_i R_j \subseteq R_{i+j}$ for all i and j. So when $R = k[x_1, \dots, x_n]$ is standard graded,

$$R_i = \bigoplus_{a_1 + \cdots + a_n = i} x_1^{a_1} \cdots x_n^{a_n}.$$

More generally, a **graded ring** is any ring that can be decomposed in pieces of this form, meaning that

$$R = \bigoplus_i R_i \quad \text{and} \quad R_i R_j \subseteq R_{i+j}.$$

The homogeneous elements of degree i are the elements in R_i. A graded R-module is an R-module M such that

$$M = \bigoplus_i M_i \quad \text{and} \quad R_i M_j \subseteq M_{i+j}.$$

A homomorphism of graded R-modules $\varphi \colon M \to N$ satisfying $\varphi(M_i) \subseteq N_{i+d}$ for all i is a **graded map** of degree d. A map of degree 0 is sometimes called **degree preserving**. Any graded map can be thought of as a map of degree 0 by shifting degrees. We write $M(-d)$ for the graded R-module that has M as the underlying R-module, but where the graded structure is given by taking $M(-d)_i = M_{i-d}$.

Note that 0 can be thought of as a homogeneous element of any degree; one sometimes declares $\deg(0) = -\infty$. An ideal I in R is a **homogeneous ideal** if it can be generated by homogeneous elements; one can show that this is equivalent to the condition

$$I = \bigoplus_i (I \cap R_i).$$

Finally, whenever I itself is homogeneous, the grading on R passes onto R/I, with

$$(R/I)_i = R_i/I_i.$$

We will be concerned with finitely generated $\mathbb{N}$-graded k-algebras R with $R_0 = k$, which are of the form $R = k[x_1, \ldots, x_n]/I$ for some homogeneous ideal I. One nice feature of such rings is that while there might be many maximal ideals, there is only one *homogeneous* maximal ideal, which is given by

$$R_+ = \bigoplus_{i>0} R_i.$$

In many ways, the behavior of such a graded ring and its unique homogeneous maximal ideal R_+ is an analogue to the behavior of a local ring R and its unique maximal ideal ideal $\mathfrak{m}$, though one always needs to provide a separate proof for the graded and local versions.

We will summarize this as follows:

Setting 1 (Local/graded setting)**.** We will say that we are in the local/graded setting, or more precisely that $(R, \mathfrak{m})$ is local/graded to indicate one of the following holds:

- Local setting: R is a noetherian local ring with $\mathfrak{m}$ its unique maximal ideal.
- Graded setting: $R = k[x_1, \ldots, x_d]/I$, where k is a field, $k[x_1, \ldots, x_d]$ has the standard grading, and I is a homogeneous ideal, and $\mathfrak{m} = R_+$ is the unique homogeneous maximal ideal of R.

In the graded setting, unless otherwise stated, any statement about modules will refer to *graded* modules, and all module homomorphisms will be assumed to be graded of degree 0.

Chapter 1

Free resolutions and the Koszul complex

1.1 Minimal free resolutions

Given an R-module M, how do we describe it? We need to know a set of generators and the relations among those generators. Going further and asking for relations among the relations (treating the relations as generators for the module of relations), and relations among the relations among the relations, and so on, we construct a free resolution for M. Free resolutions play a key role in many important constructions, and encode a lot of interesting information about our module. For example, if the module came from some geometric setting, geometric information about the module gets reflected in the free resolution.

Definition 1.1. Let M be a module over a ring R. A **projective resolution** of M is a complex of projective R-modules F_i

$$F = \cdots \longrightarrow \underset{n}{F_n} \longrightarrow \cdots \longrightarrow \underset{1}{F_1} \longrightarrow \underset{0}{F_0} \longrightarrow 0$$

such that $\mathrm{H}_i(F) = 0$ for all $i \neq 0$, together with an isomorphism $\mathrm{H}_0(F) \cong M$. When all the F_i are free, we say F is a **free resolution** for M. We will abuse notation and carelessly identify F with the corresponding augmented resolution, which is the exact sequence

$$F = \cdots \longrightarrow \underset{n}{F_n} \longrightarrow \cdots \longrightarrow \underset{1}{F_1} \longrightarrow \underset{0}{F_0} \longrightarrow M \longrightarrow 0.$$

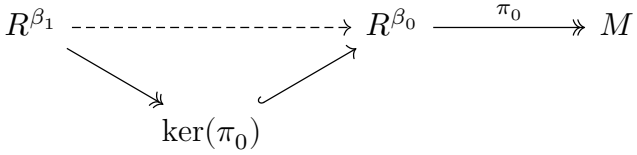
Remark 1.2. In the local setting, any finitely generated projective module must be free, by Theorem 4.55 from Homological Algebra. In fact, Kaplansky [Kap58] showed that *any* projective module over a noetherian local ring must be free. In the graded setting, one can also show (exercise!) that every bounded below graded projective must be free. Therefore, any projective resolution in the local/graded setting is in fact a free resolution.

Thus we will focus on free resolutions. We can think of a free resolution of M as an approximation of M by free modules. Since every module is a quotient of a free module, every module has a free resolution. Let us recall the construction we saw in Homological Algebra:

Construction 1.3 (Minimal free resolution)**.** Let M be a finitely generated module over R, where R is either local or graded as in our general setup. If M has β_0 many minimal generators, then we can write a surjective R-module homomorphism from R^{β_0} to M, say

$$R^{\beta_0} \xrightarrow{\pi_0} M$$

If π_0 is an isomorphism, then $M \cong R^{\beta_0}$ is a free module of rank β_0. Otherwise, π_0 has a nonzero kernel $\ker(\pi_0)$, which must also be a finitely generated module since R is noetherian. If $\ker(\pi_0)$ is minimally generated by β_1 elements, then we repeat this process and construct a surjective R-module map from R^{β_1} to $\ker(\pi_0)$, and compose it with the inclusion of $\ker(\pi_0)$ into R^{β_0}:



The elements in $\ker(\pi_0)$ are the **relations** on our chosen minimal generators for M: if M is generated by $m_1, \dots, m_{\beta_0}$, we can take π_0 to be the map sending each canonical basis element e_i in R^{β_0} to m_i, and each $(r_1, \dots, r_{\beta_0}) \in \ker(\pi_0)$ corresponds to a **relation** among the m_i, meaning

$$r_1 m_1 + \cdots + r_{\beta_0} m_{\beta_0} = 0.$$

Such relations are called **syzygies**[1] of M and the module $\ker(\pi_0)$ is the first syzygy module of M, which we will denote by $\Omega_1(M)$.

Continuing this process, we construct a free resolution for M:

$$\cdots \longrightarrow F_n \xrightarrow{\pi_n} \cdots \xrightarrow{\pi_2} F_1 \xrightarrow{\pi_1} F_0 \xrightarrow{\pi_0} M \longrightarrow 0.$$

In the local/graded setting and when M is a finitely generated (graded) module, we can choose F_i at each step to have the minimal number of generators; in that case, we say that F is a **minimal free resolution** for M.

Back in Homological Algebra, we then showed the following remarkable facts:

- Every free resolution of M has a minimal free resolution of M as a direct summand.
- Any two minimal free resolutions of M are isomorphic complexes, thus we can talk about *the* minimal free resolution of M.
- As a consequence of the previous facts, the minimal free resolution of M must have the shortest length of any resolution for M, and M has a finite resolution if and only if the minimal free resolution of M is finite.
- A free resolution F of M with differential ∂ is minimal if and only if $\partial(F) \subseteq \mathfrak{m}F$. Thus if we fix bases for all the free modules F_i, the resolution is minimal if and only if all the entries in the matrices representing ∂ have all entries in $\mathfrak{m}$.

[1] Fun fact: in astronomy, a syzygy is an alignment of three or more celestial objects.

Definition 1.4. In general, any complex (F, ∂) satisfying $\partial(F) \subseteq \mathfrak{m}F$ is called a **minimal complex**.

Definition 1.5. Let M be an R-module. A finite projective resolution

$$F = \cdots \longrightarrow 0 \longrightarrow F_c \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow 0$$

has length c if $F_c \neq 0$ and $F_i = 0$ for all $i \geqslant c$. A resolution F has infinite length if $F_i \neq 0$ for all $i \geqslant 0$. The **projective dimension** of M is

$$\operatorname{pdim}_R(M) := \inf\{c \mid M \text{ has a projective resolution of length } c\}.$$

Remark 1.6. As we noted in Remark 1.2, in the local/graded setting, any projective resolution is in fact a free resolution. Thus

$$\operatorname{pdim}_R(M) = \inf\{c \mid M \text{ has a free resolution of length } c\}.$$

Note that if a module M has a finite (minimal) free resolution, then the projective dimension of M is simply the length of the minimal free resolution for M. If the minimal free resolution of M is infinite, then $\operatorname{pdim} M = \infty$.

Remark 1.7. Note that $\operatorname{pdim}(M) = 0$ if and only if M is projective (and in the local/graded setting, free).

Definition 1.8. Consider a minimal free resolution F of M, and consider the notation in Construction 1.3. The **ith syzygy module of** M, denoted $\Omega_i(M)$, is defined to be the image of π_i or equivalently the kernel of π_{i-1}.

Note that $\Omega_i(M)$ is defined only up to isomorphism.

Remark 1.9. There are two conventions for what module $\Omega_i(M)$ corresponds to: one could instead take $\Omega_i(M) = \ker(\pi_i)$. One advantage of setting $\Omega_i(M) = \ker(\pi_i)$ is that this makes $\Omega_i(M)$ a submodule of F_i. But one advantage of the convention $\Omega_i(M) = \ker(\pi_{i-1})$ we have chosen is that the module of first syzygies of M, $\Omega_1(M)$, corresponds to the relations among the generators of M, which are the first set of relations we want to consider.

We recommend always checking the definition used in a particular reference.

Exercise 1. Show that for all R-modules M and all $i \geqslant 1$, if F if the minimal free resolution for M, then there is a natural short exact sequence

$$0 \longrightarrow \Omega_{i+1}(M) \longrightarrow F_i \longrightarrow \Omega_i(M) \longrightarrow 0.$$

Remark 1.10. Suppose that at some point when building a resolution following the procedure we described in Construction 1.3, we obtain an injective map of free modules. Then its kernel is trivial, so we obtain a finite free resolution.

The projective dimension of a finitely generated module can be infinite.

Example 1.11. Let k be a field and $R = k[x]/(x^2)$, which is a local ring with maximal ideal $\mathfrak{m} = (x)$. The residue field $k = R/\mathfrak{m}$ has infinite projective dimension: indeed, its minimal free resolution is

$$\cdots \longrightarrow R \xrightarrow{x} R \xrightarrow{x} R \xrightarrow{x} R \longrightarrow k \longrightarrow 0.$$

So even cyclic modules can have infinite projective dimension.

These invariants can give us some information about Ext and Tor, and vice-versa.

Remark 1.12. If $\operatorname{pdim}_R(M) = n$ is finite, then $\operatorname{Tor}_i^R(M, -) = 0$ and $\operatorname{Ext}_R^i(M, -) = 0$ for all $i > n$, since for all R-modules N, we can compute $\operatorname{Tor}_i^R(M, N)$ and $\operatorname{Ext}_R^i(M, N)$ using a free resolution for M of length n.

The ranks of the free modules in the minimal free resolution are particularly important invariants of M:

Definition 1.13 (Betti numbers)**.** Let F be the minimal free resolution of M. The **ith Betti number** of M is

$$\beta_i(M) = \operatorname{rank}(F_i).$$

Remark 1.14. Note that $\beta_0(M) = \mu(M) = \dim_k(M/\mathfrak{m}M)$.

Example 1.15. Let $R = k[\![x, y]\!]$ and $M = R/(x^2, xy)$. Let us write a minimal free resolution for M. First, we note that M is cyclic, so we start with

$$R \longrightarrow R/(x^2, xy) \longrightarrow 0.$$

In this case, the relations on the unique generator 1 in degree 0 are $x^2 \cdot 1 = 0$ and $xy \cdot 1 = 0$, so we proceed with

$$R^2 \xrightarrow{\begin{pmatrix} x^2 & xy \end{pmatrix}} R \longrightarrow R/(x^2, xy) \longrightarrow 0.$$

Now we need to find all relations among x^2 and xy, meaning all choices of a and b in R such that $ax^2 + bxy = 0$. We note that x is a regular element on R, so $ax^2 = -bxy \implies ax = -by$. But x and y are nonassociate irreducibles, so $a \in (y)$ and $b \in (x)$. Thus

$$\underline{y} \cdot x^2 + \underline{-x} \cdot xy = 0$$

is one of the relations we are looking for, and *all* other such relations are multiples of this one. This shows that we can continue our resolution by taking

$$R \xrightarrow{\begin{pmatrix} y \\ -x \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} x^2 & xy \end{pmatrix}} R \longrightarrow R/(x^2, xy) \longrightarrow 0.$$

Now note that R is a domain, so the leftmost map is in fact injective, and we are done. We conclude that

$$0 \longrightarrow \underset{2}{R} \xrightarrow{\begin{pmatrix} y \\ -x \end{pmatrix}} \underset{1}{R^2} \xrightarrow{\begin{pmatrix} x^2 & xy \end{pmatrix}} \underset{0}{R} \longrightarrow R/(x^2, xy) \longrightarrow 0$$

is a free resolution for $R/(x^2, xy)$. We also took as few generators at each step as possible, so this is a minimal free resolution. We can check this more precisely by noting that all the entries in our matrices are nonunits. In particular, we learn that $\operatorname{pdim}(R/(x^2, xy)) = 2$.

Example 1.16. Let $R = k[x, y, z]$ and $M = R/(xy, xz, yz)$. We claim that the minimal free resolution for M is

$$0 \longrightarrow R^2 \xrightarrow{\begin{pmatrix} z & 0 \\ -y & y \\ 0 & -x \end{pmatrix}} R^3 \xrightarrow{\begin{pmatrix} xy & xz & yz \end{pmatrix}} R \longrightarrow M \longrightarrow 0.$$

The Betti numbers of M are

$$\beta_0(M) = 1 \quad \beta_1(M) = 3 \quad \beta_2(M) = 2.$$

This is a special case of the Hilbert–Burch Theorem [Bur68], which tells us about the shape of the minimal free resolution of cyclic modules of projective dimension 2.

In the example above, we presented a minimal free resolution without justification. How could one check by hand that this is indeed the minimal free resolution? In theory, we would find the kernel of each differential map explicitly, which can be quite challenging, and then check that it does in fact match the image of the next differential. But in specific cases, we can take advantage of other facts about resolutions to determine some or all of the betti numbers, for example, which can help justifying a particular complex is in fact a resolution. As for minimality, we can simply check that all the entries in the matrices presented are in fact in the unique (homogeneous) maximal ideal, showing that the complex is minimal.

Over the course of this class, we will be discussing some helpful properties of projective dimension and betti numbers that will help us justify these things more carefully.

In the graded setting, we can take resolutions of graded free modules and graded maps. Including this graded information enhances the kind of information we can obtain from a resolution. In particular, we can consider graded betti numbers, which take into account not only the number of generators in each homological degree but also what their internal degree (R-degree) is. We will explore this in more detail in the next section.

1.2 Graded betti numbers

Let R be a standard graded k-algebra with $R_0 = k$ and homogeneous maximal ideal $\mathfrak{m} = R_+$. Let M be a graded R-module. To write a graded free resolution for M, we choose all maps to have degree 0, so that the graded free modules in each degree are sums of copies of shifts of R. We write $R(-d)$ for the R-module R but with a new grading, where

$$(R(-d))_i := R_{i-d}.$$

Definition 1.17. Let M be a graded R-module with minimal graded free resolution F. The (i,j)**th Betti number** of M, $\beta_{ij}(M)$, counts the number of generators of F_i in degree j. Thus

$$\beta_{i,j}(M) = \text{number of copies of } R(-j) \text{ in } F_i \quad \text{and} \quad F_i = \bigoplus_j R(-j)^{\beta_{ij}(M)}.$$

Remark 1.18. At each stage, the nonzero entries in the differential must be nonunits, and thus they are homogeneous elements of positive degree. Therefore, if

$$F_i = R(-d_1)^{\beta_{id_1}} \oplus \cdots \oplus R(-d_s)^{\beta_{id_s}}$$

with $d_1 \leqslant \cdots \leqslant d_s$, and $F_{i+1} \neq 0$, then the smallest possible shift in F_{i+1} is $d_1 + 1$. In particular, $\beta_{ij}(M) = 0$ for all $j < i$.

Definition 1.19. We often collect the Betti numbers of a module in its **Betti table**:

$\beta(M)$	0	1	2	$\cdots$
0	$\beta_{00}(M)$	$\beta_{11}(M)$	$\beta_{22}(M)$	
1	$\beta_{01}(M)$	$\beta_{12}(M)$	$\beta_{23}(M)$	
2	$\beta_{02}(M)$	$\beta_{13}(M)$		
$\vdots$			$\ddots$	

By convention, the entry corresponding to (i,j) in the Betti table of M contains $\beta_{i,i+j}(M)$, and *not* $\beta_{ij}(M)$. This is how Macaulay2 displays Betti tables.

Example 1.20. From the minimal resolution in Example 1.16, we can read the graded Betti numbers of M:

- $\beta_0(M) = 1$, since M is cyclic. The unique generator lives in degree 0, so $\beta_{0,0}(M) = 1$.
- $\beta_1(M) = 3$, and these three quadratic generators live in degree 2, so $\beta_{12} = 3$.
- $\beta_2(M) = 2$. These are linear syzygies on quadrics, living in degree 1+2=3, so $\beta_{23} = 2$.

Here is the graded free resolution of M:

$$0 \longrightarrow R(-3)^2 \xrightarrow{\begin{pmatrix} z & 0 \\ -y & y \\ 0 & -x \end{pmatrix}} R(-2)^3 \xrightarrow{\begin{pmatrix} xy & xz & yz \end{pmatrix}} R \longrightarrow M \longrightarrow 0.$$

Notice that the graded shifts in lower homological degrees affect all the higher homological degrees as well. For example, when we write the map in degree 2, we only need to shift the degree of each generator by 1, but since our map now lands on $R(-2)^3$, we have to bump up degrees from 2 to 3, and write $R(-3)^2$. So again we have

$$\beta_{00} = 1, \beta_{12} = 3, \text{and } \beta_{23} = 2.$$

We can now collect the graded Betti numbers of M in its Betti table:

	0	1	2
0	1	–	–
1	–	3	2

.

Example 1.21. Let k be a field, $R = k[x, y]$, and consider the ideal

$$I = (x^2, xy, y^3)$$

which has two generators of degree 2 and one of degree 3, so there are graded Betti numbers β_{12} and β_{13}. The minimal free resolution for R/I is

$$0 \longrightarrow \begin{matrix} R(-3)^1 \\ \oplus \\ R(-4)^1 \end{matrix} \xrightarrow{\begin{pmatrix} y & 0 \\ -x & y^2 \\ 0 & -x \end{pmatrix}} \begin{matrix} R(-2)^2 \\ \oplus \\ R(-3)^1 \end{matrix} \xrightarrow{\begin{pmatrix} x^2 & xy & y^3 \end{pmatrix}} R \longrightarrow R/I.$$

Thus
$$\beta_{23}(R/I) = 1 \qquad \beta_{12}(R/I) = 2$$
$$\beta_{24}(R/I) = 1 \qquad \beta_{13}(R/I) = 1$$

and the Betti table of R/I is

$\beta(M)$	0	1	2
0	1	–	–
1	–	2	1
2	–	1	1

.

Even if all we know is the Betti numbers of M, there is lots of information we can extract about M. For more about the beautiful theory of free resolutions and syzygies, see [Eis05]. For a detailed treatment of graded free resolutions, see [Pee11].

Macaulay2. In Macaulay2, given an R-module M, we can ask for its minimal free resolution by running `res M`. If I is an ideal in R, note that `res I` returns a resolution for R/I. Currently, resolutions still default to type `ChainComplexes`, but the default will soon be replaced with `Complexes`. You can ask for a resolution of type `Complexes` by loading the `Complexes package` and asking for `freeResolution M`.

More on `ChainComplexes` vs `Complexes` in Macaulay2 can be found in Appendix A.

1.3 The residue field has the largest resolution

The residue field plays an especially important role in the story of free resolutions:

Exercise 2. Let $(R, \mathfrak{m}, k)$ be a noetherian local ring, and M be a finitely generated R-module. Show that $\beta_i(M) = \dim_k \left(\operatorname{Tor}_i^R(M,k)\right) = \dim_k \left(\operatorname{Ext}^i_R(M,k)\right)$.

Remark 1.22. As a consequence of Exercise 2, we learn that

$$\operatorname{pdim}(M) = \sup\{i \mid \beta_i(M) \neq 0\} = \sup\{i \mid \operatorname{Ext}^i_R(M,k) \neq 0\} = \sup\{i \mid \operatorname{Tor}^i_R(M,k) \neq 0\}.$$

We can extend Exercise 2 to graded betti numbers once we realize that Tor and Ext of graded modules can also be given graded structures.

Remark 1.23. When R is a graded ring and M and N are graded R-modules, we can compute $\operatorname{Ext}^i_R(M,N)$ using a graded free resolution of M, and thus the Ext-modules inherit an R-graded structure.

Exercise 3. Let R be a standard graded finitely generated algebra over a field $k = R_0$ and let M be a graded R-module. Show that

$$\beta_{i,j}(M) = \text{number of copies of } R(-j) \text{ in } F_i = \dim_k \left(\operatorname{Tor}_i^R(M,k)_j\right) = \dim_k \left(\operatorname{Ext}^i_R(M,k)_{-j}\right).$$

The residue field has the largest free resolution possible.

Corollary 1.24. *Let $(R, \mathfrak{m}, k)$ be local/graded. Then for every finitely generated (graded) R-module M,*

$$\operatorname{pdim}_R(M) \leqslant \operatorname{pdim}_R(k).$$

Proof. It suffices to consider the case when $\operatorname{pdim}_R(k)$ is finite. When $i > \operatorname{pdim}_R(k)$, we must have $\operatorname{Tor}_i^R(M,k) = 0$, as noted in Remark 1.12. By Exercise 2, $\beta_i(M) = 0$ for all such i, so $\operatorname{pdim}(M) \leqslant \dim(k)$. □

Definition 1.25. Let R be a domain with fraction field Q. The **rank** of a finitely generated R-module M is defined as

$$\operatorname{rank} M := \dim_Q(M \otimes_R Q).$$

Exercise 4. Check that if M is a free module, then $\operatorname{rank}(M)$ is the free rank of M.

Exercise 5. Let M be a finitely generated module over a noetherian local domain. Show that

$$\beta_i(M) = \operatorname{rank}\left(\Omega_i(M)\right) + \operatorname{rank}\left(\Omega_{i+1}(M)\right).$$

Exercise 6. Show that if M has finite projective dimension over a domain, then

$$\sum_{i=0}^{\operatorname{pdim}(M)} (-1)^i \beta_i(M) = \operatorname{rank}(M).$$

If only we had an explicit minimal free resolution of k, maybe we could use it to say something about the minimal free resolutions of other finitely generated R-modules. With that goal in mind, we take a break from thinking about free resolutions and projective dimension to discuss a very important complex – perhaps *the* most important complex.

1.4 The Koszul complex

The Koszul complex is arguably the most important complex in commutative algebra (and beyond). It appears everywhere, and it is a very powerful yet elementary tool any homological algebraist needs in their toolbox. Every sequence of elements $x_1, \dots, x_n$ in any ring R gives rise to a Koszul complex.

Construction 1.26 (The Koszul complex)**.** The **Koszul complex** on one element $x \in R$ is the complex

$$\operatorname{kos}(x) := 0 \longrightarrow \underset{1}{R} \xrightarrow{x} \underset{0}{R} \longrightarrow 0.$$

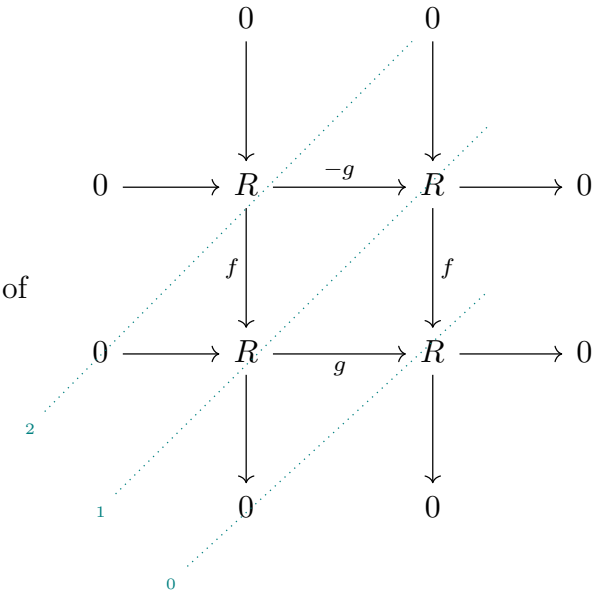
More generally, given $x_1, \dots, x_n \in R$, the **Koszul complex** with respect to $x_1, \dots, x_n$ is the complex $\operatorname{kos}(x_1, \dots, x_n)$ defined inductively as

$$\operatorname{kos}(x_1, \dots, x_n) := \operatorname{kos}(x_1, \dots, x_{n-1}) \otimes_R \operatorname{kos}(x_n).$$

You will find different sign conventions for the Koszul complex in the literature, but at the end of the day they all lead to isomorphic complexes.

Example 1.27. The Koszul complex on $f, g \in R$ is given by

$\operatorname{kos}(f,g) =$ Totalization of



which is

$$0 \longrightarrow R \xrightarrow{\binom{-g}{f}} R^2 \xrightarrow{(f \quad g)} R \longrightarrow 0.$$

In general, computing the successive tensor products is a bit tedious. Instead, we will give an alternative way to think about the Koszul complex.

Definition 1.28. The **exterior algebra** $\bigwedge M$ on an R-module M is obtained by taking the the free R-algebra $R \oplus M \oplus (M \otimes M) \oplus (M \otimes M \otimes M) \oplus \cdots$, modulo the relations $x \otimes y = -y \otimes x$ and $x \otimes x = 0$ for all $x, y \in M$. We denote the product on $\bigwedge M$ by $a \wedge b$, and see $\bigwedge M$ as a graded algebra where the homogeneous elements in degree d consist of the image of $M^{\otimes d}$. This is a **skew commutative** algebra: for all homogeneous elements a and b

$$a \wedge b = (-1)^{\deg(a)\deg(b)} b \wedge a \quad \text{and} \quad a \wedge a = 0 \text{ whenever } a \text{ has odd degree.}$$

We denote the set of all homogeneous elements of degree n by $\bigwedge^n M$. Note also that this construction is functorial: a map $f \colon M \to N$ of R-modules induces a map

$$\begin{aligned} \textstyle\bigwedge M &\xrightarrow{\quad \wedge f \quad} \textstyle\bigwedge N \\ m_1 \wedge \cdots \wedge m_s &\longmapsto f(m_1) \wedge \cdots \wedge f(m_s). \end{aligned}$$

We will primarily use this construction in the case of free modules. When $M = R^n$ with basis $e_1, \dots, e_n$, then for all $1 \leqslant s \leqslant n$

$$\wedge^s M \cong R^{\binom{n}{s}} \quad \text{with basis} \quad e_{i_1} \wedge \cdots \wedge e_{i_s} \text{ ranging over all } i_1 < i_2 < \cdots < i_s.$$

Definition 1.29 (The Koszul complex, again)**.** Let $x_1, \dots, x_n$ be elements in R. The **Koszul complex** on $x_1, \dots, x_n$ is the complex

$$\operatorname{kos}(x_1, \dots, x_n) := \; 0 \longrightarrow \textstyle\bigwedge^n R^n \longrightarrow \bigwedge^{n-1} R^n \longrightarrow \cdots \longrightarrow \bigwedge^1 R^n \longrightarrow R \longrightarrow 0$$

with differential

$$\partial(e_{i_1} \wedge \cdots \wedge e_{i_s}) = \sum_{1 \leqslant p \leqslant s} (-1)^{p-1} x_{i_p} e_{i_1} \wedge \cdots \wedge \widehat{e_{i_p}} \wedge \cdots \wedge e_{i_s}.$$

Exercise 7. Show that d as defined above is indeed a differential, meaning $d^2 = 0$.

Exercise 8. Check that our two definitions of the Koszul complex coincide.

Example 1.30. In the case of two elements, say x and y in R,

$$\operatorname{kos}(x, y) = \; 0 \longrightarrow \wedge^2 R^2 \longrightarrow \wedge^1 R^2 \longrightarrow R \longrightarrow 0$$

with $\partial(e_1) = x$, $\partial(e_2) = y$, $\partial(e_1 \wedge e_2) = xe_2 - ye_1$, so

$$\operatorname{kos}(x, y) = \; 0 \longrightarrow R \xrightarrow{\begin{pmatrix} -y \\ x \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} x & y \end{pmatrix}} R \longrightarrow 0.$$

Exercise 9. Write the Koszul complex on 3 elements f_1, f_2, f_3.

The previous exercise is must easier to do using the exterior algebra description of the Koszul complex.

Remark 1.31. You will find different sign conventions for the Koszul complex in the literature, but at the end of the day they all lead to isomorphic complexes.

Definition 1.32. Let M be an R-module and let $x_1, \dots, x_n \in R$. The **Koszul complex** on M with respect to $x_1, \dots, x_n$ is the complex

$$\mathrm{kos}(x_1, \dots, x_n; M) := \mathrm{kos}(x_1, \dots, x_n) \otimes_R M.$$

Remark 1.33. In general, the Koszul complex $\mathrm{kos}(x_1, \dots, x_n; M)$ looks like

$$0 \longrightarrow M \longrightarrow M^n \longrightarrow \cdots \longrightarrow M^{\binom{n}{i}} \longrightarrow \cdots \longrightarrow M^n \longrightarrow M \longrightarrow 0$$

and the nonzero entries in the differential matrices consist of our elements $x_1, \dots, x_n$ with carefully chosen some signs. The left most map, in degree n, is given by the matrix

$$\begin{pmatrix} x_1 \\ -x_2 \\ x_3 \\ \vdots \\ (-1)^{n+1} x_n \end{pmatrix}$$

and the rightmost map is

$$\begin{pmatrix} x_1 & x_2 & x_3 & \cdots & x_n \end{pmatrix}.$$

Our exterior algebra description of the Koszul complex has the advantage that it indicates a bonus structure on our complex: it is also an algebra. In fact, this is the first big example of a DG algebra. While we will not have the chance to explore this further, this DG algebra structure on the Koszul complex plays a major role in commutative algebra.

We can also take advantage of this added structure to note that the Koszul complex is a functorial construction: given $\underline{x} = x_1, \dots, x_n \in R$ and $\underline{y} = y_1, \dots, y_m \in R$, any commutative diagram of R-module homomorphisms

$$\begin{array}{ccc} R^n & \xrightarrow{\begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix}} & R \\ {\scriptstyle \varphi_1}\downarrow & & \downarrow{\scriptstyle \varphi_0} \\ R^m & \xrightarrow[\begin{pmatrix} y_1 & \cdots & y_m \end{pmatrix}]{} & R \end{array}$$

extends to a map of complexes $\mathrm{kos}(\underline{x}) \longrightarrow \mathrm{kos}(\underline{y})$, by taking

$$\varphi_s\left(e_{i_1} \wedge \cdots \wedge e_{i_s}\right) = \varphi_1(e_{i_1}) \wedge \cdots \wedge \varphi_1(e_{i_1}).$$

The fact that this map commutes with taking the differential is an immediate consequence of the definition, which we leave as an exercise.

Notation 1. Given elements $x_1, \dots, x_n$ in a ring R, we often write $\underline{x} = x_1, \dots, x_n$.

The homology of the Koszul complex has some nice properties.

Definition 1.34. Let M be an R-module and $x_1, \dots, x_n \in R$. The **ith Koszul homology** module of M with respect to $x_1, \dots, x_d$, also called the **Koszul homology** on $x_1, \dots, x_d$ with coefficients in M, is the R-module

$$\mathrm{H}_i(x_1, \dots, x_d; M) := \mathrm{H}_i(\mathrm{kos}(x_1, \dots, x_d; M)).$$

Exercise 10. Let $(R, \mathfrak{m}, k)$ be a noetherian local ring, and let $\underline{x}$ and $\underline{y}$ be two minimal generating sets for the same ideal. Show that $\mathrm{kos}(\underline{x})$ and $\mathrm{kos}(\underline{y})$ are isomorphic complexes.

Theorem 1.35. *Let R be a ring, $I = (x_1, \dots, x_n)$ be an ideal, and M an R-module.*

1) $\mathrm{H}_i(\underline{x}; M) = 0$ *whenever* $i < 0$ *or* $i > n$.

2) $\mathrm{H}_0(\underline{x}; M) = M/IM$.

3) $\mathrm{H}_n(\underline{x}; M) = (0 :_M I) = \mathrm{ann}_M(I)$.

4) Every Koszul homology module $\mathrm{H}_i(\underline{x}; M)$ *is killed by* $\mathrm{ann}_R(M)$.

5) Every Koszul homology module $\mathrm{H}_i(\underline{x}; M)$ *is killed by* I.

6) If M is a noetherian R-module, so is $\mathrm{H}_i(\underline{x}; M)$ *for every i.*

7) For every i, $\mathrm{H}_i(\underline{x}; -)$ *is a covariant additive functor R-**Mod** $\longrightarrow$ R-**Mod**.*

8) Every short exact sequence of R-modules

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

gives rise to a long exact sequence on Koszul homology,

$$\cdots \longrightarrow \mathrm{H}_1(\underline{x}; C) \longrightarrow \mathrm{H}_0(\underline{x}; A) \longrightarrow \mathrm{H}_0(\underline{x}; B) \longrightarrow \mathrm{H}_0(\underline{x}; C) \longrightarrow 0.$$

Proof.

1) Immediate from the definition, since the Koszul complex is only nonzero in homological degrees 0 through n.

2) Remark 1.33 tells us that

$$\mathrm{H}_0(\underline{x}; M) = \mathrm{coker}\left(M \xrightarrow{\begin{pmatrix} x_1 & x_2 & x_3 & \cdots & x_n \end{pmatrix}} M\right) = M/IM.$$

3) Remark 1.33 above tells us that

$$\begin{aligned} \mathrm{H}_n(\underline{x}; M) &= \ker\left(M \xrightarrow{\begin{pmatrix} x_1 & -x_2 & x_3 & \cdots & (-1)^{n+1}x_n \end{pmatrix}^T} M^n\right) \\ &= \{m \in M \mid rx_1 = rx_2 = \cdots = rx_n = 0\} \\ &= (0 :_M I). \end{aligned}$$

4) In each homological degree, the Koszul complex is simply a direct sum of copies of M. So the modules in the complex $\operatorname{kos}(\underline{x}; M)$ are themselves already killed by $\operatorname{ann}_R(M)$, before we even take homology.

5) We are going to show something stronger: we will show that for all $a \in I$, multiplication by a on $\operatorname{kos}(\underline{x}; M)$ is nullhomotopic, which proves that multiplication by a is the zero map in $\operatorname{H}_i(\underline{x}; M)$. In fact, it is sufficient to show that multiplication by a is nullhomotopic on $\operatorname{kos}(\underline{x})$, since additive functors preserve the homotopy relation. To do this, we will explicitly use the multiplicative structure of the Koszul complex given by our description of the Koszul complex via exterior powers. Given $a \in I = (x_1, \dots, x_n)$, write $a = a_1x_1 + \cdots + a_nx_n$. Consider the map

$$s_a \colon \operatorname{kos}(\underline{x}) \longrightarrow \Sigma^{-1} \operatorname{kos}(\underline{x})$$

given by multiplication by $a_1e_1 \wedge \cdots \wedge a_ne_n$, meaning

$$s_a(e_{i_1} \wedge \cdots \wedge e_{i_t}) = \sum_{j=1}^{n} a_j e_j \wedge e_{i_1} \wedge \cdots \wedge e_{i_t}.$$

Now we claim this map s_a is a nullhomotopy for the map of complexes $\operatorname{kos}(\underline{x}) \longrightarrow \operatorname{kos}(\underline{x})$ given by multiplication by a in every component. To check that, it is sufficient to check this on basis elements, that is, we need only to check that

$$s_a\partial\left(e_{i_1} \wedge \cdots \wedge e_{i_t}\right) + \partial s_a\left(e_{i_1} \wedge \cdots \wedge e_{i_t}\right) = ae_{i_1} \wedge \cdots \wedge e_{i_t}.$$

Indeed, we have

$$\begin{aligned} s_a\partial\left(e_{i_1} \wedge \cdots \wedge e_{i_t}\right) &= s_a\left(\sum_{k=1}^{t} (-1)^{k+1} x_k e_{i_1} \wedge \cdots \wedge \widehat{e_{j_k}} \wedge \cdots \wedge e_{i_t}\right) \\ &= \sum_{j=1}^{n}\sum_{k=1}^{t} (-1)^{k+1} a_j x_k e_j \wedge e_{i_1} \wedge \cdots \wedge \widehat{e_{j_k}} \wedge \cdots \wedge e_{i_t} \end{aligned}$$

and

$$\begin{aligned} \partial s_a\left(e_{i_1} \wedge \cdots \wedge e_{i_t}\right) &= \partial\left(\sum_{j=1}^{n} a_j e_j \wedge e_{i_1} \wedge \cdots \wedge e_{i_t}\right) \\ &= \sum_{j=1}^{n}\sum_{k=1}^{t} (-1)^{k+2} a_j e_j \wedge e_{j_1} \wedge \cdots \wedge \widehat{e_{i_k}} \wedge \cdots \wedge e_{i_s} + \sum_{j=1}^{n} a_j x_j e_{i_1} \wedge \cdots \wedge e_{i_t} \\ &= -s_a d\left(e_{i_1} \wedge \cdots \wedge e_{i_t}\right) + \sum_{j=1}^{n} a_j x_j e_{i_1} \wedge \cdots \wedge e_{i_t} \end{aligned}$$

and since $a_1x_1 + \cdots + a_nx_n = a$, we conclude that

$$(s_a\partial + \partial s_a)\left(e_{i_1} \wedge \cdots \wedge e_{i_t}\right) = ae_{i_1} \wedge \cdots \wedge e_{i_t}.$$

6) If M is noetherian, then so is M^k for any k, as well as any submodules of M^k and any of their quotients, by Theorem 1.53 from Commutative Algebra I. Each $\mathrm{H}_i(\underline{x}; M)$ is a subquotient of a direct sum of copies of M, so it must be noetherian.

7) Given an R-module homomorphism $f\colon M \to N$, we get an induced map

$$\mathrm{kos}(f)\colon \mathrm{kos}(\underline{x}; M) \longrightarrow \mathrm{kos}(\underline{x}; N)$$

by taking $\mathrm{kos}(\underline{x}) \otimes f$, so we set $\mathrm{H}_i(\underline{x}; f) = \mathrm{H}_i(\mathrm{kos}(\underline{x}) \otimes f)$. It is immediate to see that this assignment takes the identity on M to itself, and we leave it as an exercise to check that this definition preserves composition of homomorphisms.

8) Given any complex of free R-modules F, tensoring with F is an exact functor from R-modules to $\mathrm{Ch}(R)$. In particular, this applies to $\mathrm{kos}(\underline{x})$, so we get a short exact sequence of complexes

$$0 \longrightarrow \mathrm{kos}(\underline{x}) \otimes_R A \longrightarrow \mathrm{kos}(\underline{x}) \otimes_R B \longrightarrow \mathrm{kos}(\underline{x}) \otimes_R C \longrightarrow 0$$

The resulting long exact sequence in homology, as in Theorem 2.39 from Homological Algebra, is the long exact sequence we are looking for. □

In general, the koszul homology functor is neither left nor right exact; we will see examples later in Example 1.54.

Remark 1.36. Following our iterative definition of the Koszul complex, where

$$\mathrm{kos}(x_1, \dots, x_{n+1}; M) = \mathrm{kos}(x_1, \dots, x_n; M) \otimes \mathrm{kos}(x_{n+1}),$$

set $C := \mathrm{kos}(x_1, \dots, x_n; M)$, and note that by definition of tensor product of complexes we have

$$[\mathrm{kos}(x_1, \dots, x_{n+1}; M)]_i = C_{i-1} \otimes_R R \oplus C_i \otimes_R R \cong C_{i-1} \oplus C_i.$$

Let us explicitly write down the differential on $\mathrm{kos}(x_1, \dots, x_{n+1}; M)$ in terms of the differential on $\mathrm{kos}(x_1, \dots, x_n; M)$. Given $a \in C_{i-1}$, $b \in C_i$, $r \in R$ (in homological degree 1) and $s \in R$ (in homological degree 0), our differential is

$$\partial(a \otimes r + b \otimes s) = \partial(a) \otimes r + (-1)^{i-1} a \otimes (x_{n+1} r) + \partial(b) \otimes s + (-1)^i b \otimes 0,$$

so

$$\partial_i := \begin{pmatrix} \partial_C & 0 \\ (-1)^{i-1} x_{n+1} & \partial_C \end{pmatrix} : \begin{matrix} C_{i-1} \\ \oplus \\ C_i \end{matrix} \to \begin{matrix} C_{i-2} \\ \oplus \\ C_{i-1}. \end{matrix}$$

Notice that this is exactly[2] the cone of the map $\mathrm{kos}(x_1, \dots, x_n; M) \xrightarrow{x_{n+1}} \mathrm{kos}(x_1, \dots, x_n; M)$ given by multiplication by x_{n+1} in every degree. This cone comes together with a short exact sequence

$$0 \longrightarrow \mathrm{kos}(x_1, \dots, x_n; M) \longrightarrow \mathrm{kos}(x_1, \dots, x_{n+1}; M) \longrightarrow \Sigma^{-1} \mathrm{kos}(x_1, \dots, x_n; M) \longrightarrow 0.$$

[2] Up to the sign convention differences we discussed in Theorem 6.20 from Homological Algebra.

This short exact sequence gives rise to a long exact sequence in homology, as described in Theorem 6.23 from Homological Algebra, where (up to sign) the connecting homomorphism is simply the map in homology induced by multiplication by x_{n+1}. Here is the degree n piece of that long exact sequence:

$$\cdots \longrightarrow \mathrm{H}_i(x_1,\dots,x_{n+1};M) \longrightarrow \mathrm{H}_{i-1}(x_1,\dots,x_n;M) \xrightarrow{x_{n+1}} \mathrm{H}_{i-1}(x_1,\dots,x_n;M) \longrightarrow \cdots.$$

Let us look at

$$\mathrm{H}_i(x_1,\dots,x_n;M) \xrightarrow{x_{n+1}} \mathrm{H}_i(x_1,\dots,x_n;M) \xrightarrow{\varphi} \mathrm{H}_i(x_1,\dots,x_{n+1};M) \xrightarrow{\psi} \mathrm{H}_{i-1}(x_1,\dots,x_n;M).$$

By exactness,

$$\begin{aligned}
&\ker\left(\mathrm{H}_i(x_1,\dots,x_n;M) \xrightarrow{\varphi} \mathrm{H}_i(x_1,\dots,x_{n+1};M)\right)\\
&= \mathrm{im}(\mathrm{H}_i(x_1,\dots,x_n;M) \xrightarrow{x_{n+1}} \mathrm{H}_i(x_1,\dots,x_n;M))\\
&= x_{n+1}\cdot \mathrm{H}_i(x_1,\dots,x_n;M).
\end{aligned}$$

By the First Isomorphism Theorem, φ induces an inclusion $\overline{\varphi}$

$$\frac{\mathrm{H}_i(x_1,\dots,x_n;M)}{x_{n+1}\cdot \mathrm{H}_i(x_1,\dots,x_n;M)} \xhookrightarrow{\overline{\varphi}} \mathrm{H}_i(x_1,\dots,x_{n+1};M)$$

with $\mathrm{im}(\overline{\varphi}) = \mathrm{im}(\varphi)$. Now we use the First Isomorphism Theorem and exactness repeatedly:

$$\begin{aligned}
\mathrm{coker}(\overline{\varphi}) &= \frac{\mathrm{H}_i(x_1,\dots,x_{n+1};M)}{\mathrm{im}(\varphi)} && \text{by definition}\\
&= \frac{\mathrm{H}_i(x_1,\dots,x_{n+1};M)}{\ker(\psi)} && \text{by exactness}\\
&\cong \mathrm{im}(\psi) && \text{by the First Iso Theorem}\\
&= \mathrm{im}\left(\mathrm{H}_i(x_1,\dots,x_{n+1};M) \xrightarrow{\psi} \mathrm{H}_{i-1}(x_1,\dots,x_n;M)\right)\\
&= \ker\left(\mathrm{H}_{i-1}(x_1,\dots,x_n;M) \xrightarrow{x_{n+1}} \mathrm{H}_{n-1}(x_1,\dots,x_n;M)\right) && \text{by exactness}\\
&= \mathrm{ann}_{\mathrm{H}_{i-1}(x_1,\dots,x_n;M)}(x_{n+1}).
\end{aligned}$$

And finally, we get the following short exact sequences induced by $\overline{\varphi}$:

$$0 \longrightarrow \frac{\mathrm{H}_i(x_1,\dots,x_n;M)}{x_{n+1}\cdot \mathrm{H}_i(x_1,\dots,x_n;M)} \longrightarrow \mathrm{H}_i(x_1,\dots,x_{n+1};M) \longrightarrow \mathrm{ann}_{\mathrm{H}_{i-1}(x_1,\dots,x_n;M)}(x_{n+1}) \longrightarrow 0.$$

Definition 1.37. Let M be an R-module and consider a sequence of nonunit elements $\underline{f} \in R$. The **ith Koszul cohomology** on $\underline{f}$ with coefficients in M, also called the **Koszul cohomology** on M with respect to $\underline{f}$, is the R-module

$$\mathrm{kos}^i(\underline{f};M) := \mathrm{H}^i(\mathrm{Hom}_R(\mathrm{kos}(\underline{f}),M)).$$

We write $\mathrm{kos}^i(\underline{f}) := \mathrm{H}^i(\mathrm{Hom}_R(\mathrm{kos}(\underline{f}),R))$.

However, this adds nothing new to the story: in fact, the Koszul complex is self-dual.

Theorem 1.38. *Let R be any ring and let $\underline{x} = x_1, \dots, x_n \in R$. The Koszul complex* $\operatorname{kos}(\underline{x})$ *is isomorphic to its dual* $\operatorname{Hom}_R(\operatorname{kos}(\underline{x}), R)$*. More generally,*

$$\operatorname{kos}(\underline{x}; M) \cong \operatorname{Hom}_R(\operatorname{kos}(\underline{x}), M).$$

In particular,

$$\mathrm{H}_i(\underline{x}; M) \cong \mathrm{H}^{n-i}(\underline{x}; M).$$

Proof sketch. In general, for any two R-modules M and N, the natural isomorphism

$$R \otimes_R M \cong M$$

leads to a homomorphism

$$\operatorname{Hom}_R(N, R) \otimes_R M \xrightarrow{\psi} \operatorname{Hom}_R(N, M)$$

that sends each simple tensor $f \otimes m$ to the R-module homomorphism $N \to M$ given by

$$n \mapsto f(n) \otimes m.$$

We leave the details to the reader. When N is a finitely generated free R-module, this map ψ is an isomorphism (exercise!), so for all $d \geqslant 1$

$$\operatorname{Hom}_R(R^d, R) \otimes_R M \cong \operatorname{Hom}_R(R^d, M).$$

This isomorphism is natural, and thus induces an isomorphism

$$\operatorname{Hom}_R(\operatorname{kos}(\underline{x}, R)) \otimes M \cong \operatorname{Hom}_R(\operatorname{kos}(\underline{x}, M).$$

Thus it suffices to construct an isomorphism

$$\operatorname{kos}(\underline{x}) \xrightarrow{\omega} \operatorname{Hom}_R(\operatorname{kos}(\underline{x}, R)).$$

Such a map corresponds to a commutative diagram

$$\begin{array}{ccccccccccc}
\operatorname{kos}(\underline{x}): & 0 \longrightarrow & \bigwedge^n R^n & \longrightarrow & \bigwedge^{n-1} R^n & \longrightarrow \cdots \longrightarrow & \bigwedge^1 R^n & \longrightarrow & R & \longrightarrow 0 \\
\downarrow{\scriptstyle w} & & \downarrow{\scriptstyle w_n} & & \downarrow{\scriptstyle w_{n-1}} & & \downarrow{\scriptstyle w_1} & & \downarrow{\scriptstyle w_0} & \\
\operatorname{Hom}_R(\operatorname{kos}(\underline{x}), R): & 0 \longrightarrow & R & \longrightarrow & \bigwedge^1 R^n & \longrightarrow \cdots \longrightarrow & \bigwedge^{n-1} R^n & \longrightarrow & R & \longrightarrow 0.
\end{array}$$

For each subset $I = \{i_1, \dots, i_d\} \subseteq \{1, \dots, n\}$, write e_I for the homological degree d basis element of $\operatorname{kos}(\underline{x})$ given by

$$e_I = e_{i_1} \wedge \cdots \wedge e_{i_d} \in \bigwedge^i R^n.$$

We define $\omega_d(e_I) \in \operatorname{Hom}_R(\bigwedge^{n-i} R^n, R)$ to be the map given by

$$e_J \mapsto \begin{cases} e_I \wedge e_J & \text{if } I \cap J = \emptyset \\ 0 & \text{if } I \cap J \neq \emptyset. \end{cases}$$

We leave it as an exercise to check that this rule determines an isomorphism in each homological degree, and that the diagram commutes, proving the isomorphism between the Koszul complex and its dual. The isomorphism $\mathrm{H}_i(\underline{x}; M) \cong \mathrm{H}^{n-i}(\underline{x}; M)$ is a trivial consequence. □

1.5 Regular sequences

The Koszul complex is closely tied to regular elements and regular sequences.

Definition 1.39. Let R be a ring and M be an R-module. An element $r \in R$ is **regular** on M if $rM \neq M$ and for any $m \in M$

$$rm = 0 \Rightarrow m = 0.$$

More generally, a sequence of elements $x_1, \dots, x_n$ is a **regular sequence on** M if

- $(x_1, \dots, x_n)M \neq M$, and
- for each i, the element x_i is regular on $M/(x_1, \dots, x_{i-1})M$.

When $M = R$, we drop the *on* M and say r is regular or $x_1, \dots, x_n$ is a regular sequence.

Remark 1.40. Suppose $(x_1, \dots, x_n)M \neq M$. Note that x_i is regular on $M/(x_1, \dots, x_{i-1})M$ if and only if

$$((x_1, \dots, x_{i-1})M :_M x_i) = (x_1, \dots, x_{i-1})M.$$

Remark 1.41. When $(R, \mathfrak{m}, k)$ is a noetherian local ring and $M \neq 0$ is finitely generated R-module, NAK[3] gives us $(x_1, \dots, x_n)M \neq M$ automatically for all $\underline{x} = x_1, \dots, x_n \in \mathfrak{m}$.

Example 1.42. Consider the polynomial ring $R = k[x_1, \dots, x_n]$ in n variables over a field k. The variables $x_1, \dots, x_n$ for a regular sequence on R.

Example 1.43. Let k be a field and $R = k[x, y, z]$. The sequence xy, xz is not regular on R, since xz kills y on $R/(xy)$.

The order we write the elements in matters.

Example 1.44. Let k be a field and $R = k[x, y, z]$. We claim that $x, (x-1)y, (x-1)z$ is a regular sequence, while $(x-1)y, (x-1)z, x$ is not.

For the first claim, note that over $R/(x)$, we have $(x-1)y = -y$ and $(x-1)z = -z$. Since $x, -y, -z$ is a regular sequence on R, then so is $x, (x-1)y, (x-1)z$. In contrast, over $R/((x-1)y)$, the elements y and $(x-1)z$ are nonzero, but

$$(x-1)z \cdot y = 0.$$

In particular, $(x-1)z$ is not regular on $R/((x-1)y)$, and $(x-1)y, (x-1)z, x$ is not a regular sequence.

Remark 1.45. Suppose that $r \in R$ is such that $rM \neq M$. We claim that r is regular on M if and only if the first koszul homology vanishes, $\mathrm{H}_1(r; M) = 0$. Indeed,

$$\mathrm{H}_1(r; M) = \ker(M \xrightarrow{r} M) = (0 :_M r),$$

and by definition, r is regular on M if and only if $(0 :_M r) = 0$.

[3] Theorem 5.32 and Theorem 5.39 from Commutative Algebra I, the latter in the graded case.

In fact, the Koszul complex on a regular sequence is exact in all positive degrees.

Theorem 1.46. *If $\underline{x} = x_1, \dots, x_n \in R$ is a regular sequence on the R-module M, then $\mathrm{H}_i(\underline{x}; M) = 0$ for all $i > 0$.*

Proof. We proceed by induction on the length of the sequence, noting that the case $n = 1$ is Remark 1.45.

Suppose that for some $n \geqslant 1$, all regular sequences $\underline{y}$ on M of length n have the property that $\mathrm{H}_i(\underline{y}; M) = 0$ for all $i > 0$. Let $\underline{x} = x_1, \dots, x_{n+1}$ be a regular sequence on M. Then $x_1, \dots, x_n$ is also a regular sequence on M, so by induction hypothesis

$$\mathrm{H}_i(x_1, \dots, x_n; M) = 0 \quad \text{for all } i > 0.$$

We saw in Remark 1.36 that viewing $\mathrm{kos}(\underline{x}; M)$ as the cone of multiplication by x_{n+1} on $\mathrm{kos}(x_1, \dots, x_n; M)$ gives us a long exact sequence in homology

$$\mathrm{H}_i(x_1, \dots, x_n; M) \xrightarrow{\varphi} \mathrm{H}_i(x_1, \dots, x_{n+1}; M) \xrightarrow{\psi} \mathrm{H}_{i-1}(x_1, \dots, x_n; M).$$

For all $i \geqslant 2$, the two homologies on the left and right vanish, by induction hypothesis, and thus the middle homology must vanish as well. We conclude that $\mathrm{H}_i(x_1, \dots, x_{n+1}; M) = 0$ for all $i > 1$.

Moreover, Remark 1.36 also gave us the short exact sequence

$$0 \longrightarrow \frac{\mathrm{H}_1(x_1,\dots,x_i;M)}{x_{i+1}\cdot \mathrm{H}_1(x_1,\dots,x_i;M)} \longrightarrow \mathrm{H}_1(x_1, \dots, x_{i+1}; M) \longrightarrow \mathrm{ann}_{\mathrm{H}_0(x_1,\dots,x_i;M)}(x_{i+1}) \longrightarrow 0.$$

By induction hypothesis, $\mathrm{H}_1(x_1, \dots, x_n; M) = 0$. Since x_{n+1} is regular on

$$M/(x_1, \dots, x_n)M = \mathrm{H}_0(x_1, \dots, x_n; M),$$

we must have

$$\mathrm{ann}_{\mathrm{H}_0(x_1,\dots,x_n;M)}(x_{n+1}) = 0.$$

Therefore, in the last short exact sequence above only the middle term remains, which gives us $\mathrm{H}_1(x_1, \dots, x_{n+1}; M) = 0$. □

Corollary 1.47. *If $x_1, \dots x_n$ is a regular sequence on R, then the Koszul complex on $x_1, \dots, x_n$ is a free resolution for $R/(x_1, \dots, x_n)$. Moreover, if $(R, \mathfrak{m}, k)$ is local/graded, then $\mathrm{kos}(x_1, \dots, x_n)$ is a minimal free resolution for $R/(x_1, \dots, x_n)$.*

Proof. By Theorem 1.46, $P = \mathrm{kos}(x_1, \dots, x_n)$ has $\mathrm{H}_i(P) = 0$ for all $i > 0$. This is a complex of free modules, and thus a free resolution of $\mathrm{H}_0(P)$, which by Theorem 1.35 is $R/(x_1, \dots, x_n)$. Finally, in the local/graded case, the assumption that $\underline{x}$ is a regular sequence guarantees that $x_i \in \mathfrak{m}$ for all i. Since all the nonzero entries in the differentials (under the standard basis for the Koszul complex) are of the form $\pm x_i$, we conclude that the resolution must be minimal. □

There are three big theorems of Hilbert's every commutative algebraist must know: Hilbert's Basis Theorem, Hilbert's Nullstellensatz, and Hilbert's Syzygy Theorem. We are finally ready to prove the third.

Theorem 1.48 (Hilbert's Syzygy Theorem)**.** *Every finitely generated graded module M over a polynomial ring $R = k[x_1, \dots, x_n]$ over a field k has finite projective dimension. In fact,* $\operatorname{pdim}(M) \leqslant n$.

Proof. By Corollary 1.47, the Koszul complex on the regular sequence $x_1, \dots, x_n$ is a minimal free resolution for $k = R/(x_1, \dots, x_n)$, so $\operatorname{pdim}(k) = n$. By Corollary 1.24, every finitely generated R-module M has

$$\operatorname{pdim}(M) \leqslant \operatorname{pdim}(k) = n. \qquad \square$$

It is natural to ask if Theorem 1.46 has a converse; over a nice enough ring, the answer is yes: the vanishing of Koszul homology does characterize regular sequences. In fact, more is true: the first Koszul homology completely determines whether we have a regular sequence.

Theorem 1.49. *Let $(R, \mathfrak{m}, k)$ be local/graded. Let $M \neq 0$ be a finitely generated R-module, and $\underline{x} = x_1, \dots, x_n \in \mathfrak{m}$. In the graded case, we assume that M is graded and $x_1, \dots, x_n$ are all homogeneous. The following are equivalent:*

1) $\mathrm{H}_i(\underline{x}; M) = 0$ *for all* $i > 0$.

2) $\mathrm{H}_1(\underline{x}; M) = 0$.

3) $\underline{x}$ *is a regular sequence on* M.

Proof. The implication (a) ⇒ (b) is obvious, and (c) ⇒ (a) is Theorem 1.49. We will finish the proof by showing (b) ⇒ (c), which we will do by induction on the length n of $\underline{x}$.

The case $n = 1$ is Remark 1.45, so suppose that the statement holds for all sequences of length n for some $n \geqslant 1$. Now Remark 1.36 gives us the short exact sequence

$$0 \longrightarrow \frac{\mathrm{H}_1(x_1,\dots,x_n;M)}{x_{n+1}\cdot \mathrm{H}_1(x_1,\dots,x_n;M)} \longrightarrow \mathrm{H}_1(x_1, \dots, x_{n+1}; M) \longrightarrow \operatorname{ann}_{\mathrm{H}_0(x_1,\dots,x_n;M)}(x_{n+1}) \longrightarrow 0.$$

By assumption, $\mathrm{H}_1(x_1, \dots, x_n; M) = 0$, so exactness tells us all the terms in the short exact sequence above must vanish.

The vanishing of the left term gives us $x_{n+1} \cdot \mathrm{H}_1(x_1, \dots, x_n; M) = \mathrm{H}_1(x_1, \dots, x_n; M)$. But $\mathrm{H}_1(x_1, \dots, x_n; M)$ is a finitely generated R-module by Theorem 1.35, and $x_n \in \mathfrak{m}$, so $\mathrm{H}_1(x_1, \dots, x_n; M) = 0$ by NAK.[4] By induction hypothesis, $x_1, \dots, x_n$ is a regular sequence on M. Note moreover that NAK also guarantees that $M/(\underline{x})M \neq 0$.

Finally, the vanishing of the rightmost term in our short exact sequence gives us

$$\operatorname{ann}_{\mathrm{H}_0(x_1,\dots,x_n;M)}(x_{n+1}) = 0.$$

By definition, this means that x_{n+1} is regular on $\mathrm{H}_0(x_1, \dots, x_n; M) = M/(x_1, \dots, x_n)M$. We conclude that $\underline{x}$ is a regular sequence on M. $\square$

[4] Theorem 5.32 and Theorem 5.39 from Commutative Algebra I, the latter in the graded case.

A corollary of Theorem 1.49 is that in a local/graded ring, the order of the elements in a regular sequence does not matter.

Corollary 1.50. *Let $(R, \mathfrak{m}, k)$ be local/graded. Let M be a finitely generated R-module, and consider $\underline{x} = x_1, \dots, x_n \in \mathfrak{m}$. In the graded case, we assume that M is graded and $x_1, \dots, x_n$ are all homogeneous. If the sequence $\underline{x}$ is regular on M, then so is any of its permutations.*

Proof. If $x_1, \dots, x_n$ is a regular sequence, then Theorem 1.46 gives us

$$\mathrm{H}_i(x_1, \dots, x_n; M) = 0 \quad \text{for all } i > 0.$$

By Exercise 10, the Koszul homology on $\underline{x}$ agrees with the Koszul homology on any permutation of $\underline{x}$, which must then also vanish. By Theorem 1.49, any permutation of $\underline{x}$ is a regular sequence. □

In fact, we can extend this to any ring and any module under a reasonable assumption.

Lemma 1.51. *Let R be a ring and M an R-module. If x, y is a regular sequence on M and y is regular on M, then y, x is a regular sequence on M.*

Proof. Suppose that $xm = yn$ for some $m, n \in M$. Since x, y is a regular sequence on M and $yn \in (x)M$, we must have $n \in (x)M$, so there exists some $w \in M$ such that $n = xw$. But then $xm = yn = xyw$. Since x is regular on M, we conclude that $m = yw$, so $m \in (y)M$. In particular, this shows that x is regular on $M/(y)M$. □

Lemma 1.52. *Let $(R, \mathfrak{m})$ be a noetherian local ring. If $x_1, \dots, x_n$ is a regular sequence on M, then so is $x_1^{a_1}, \dots, x_n^{a_n}$ for any integers $a_i \geqslant 1$.*

Proof. Let $n = 1$. Suppose $x = x_1$ is a regular element on M. Given a nonzero $m \in M$ and $a \geqslant 1$,

$$x^a m = 0 \implies x x^{a-1} m = 0.$$

Since x is regular on M, we must have $x^{a-1}m = 0$. Repeating this $a - 1$ times, we conclude that $xm = 0$, and $m = 0$.

Now consider any $n \geqslant 1$. Since x_n is a regular sequence on $M/(x_1, \dots, x_{n-1})$ by the case of a sequence of length 1 we can now say $x_n^{a_n}$ is regular on $M/(x_1, \dots, x_{n-1})$. Therefore, $x_1, \dots, x_{n-1}, x_n^{a_n}$ is regular on M. By Corollary 1.50, we are allowed to permute the elements in our sequence. Now switch the order and repeat the argument with each x_i, until we conclude that $x_1^{a_1}, \dots, x_n^{a_n}$ is also regular on M. □

Using all that we have learned about the koszul complex, we can now give examples showing that koszul homology is neither left exact nor right exact.

Example 1.53. Let $R = k[x]$ and consider $M = R/(x^2)$. Note that x is regular on R but not on M, so

$$\mathrm{H}_1(x; R) = 0 \quad \text{and} \quad \mathrm{H}_1(x; M) \neq 0.$$

On the other hand, the canonical projection $R \twoheadrightarrow M$ is surjective, while the induced map on the first koszul homology

$$0 = \mathrm{H}_1(x; R) \to \mathrm{H}_1(x; M) \neq 0$$

cannot be surjective. Thus $\mathrm{H}_1(x; -)$ is not right exact.

Example 1.54. Let $R = k[x]$ and note that x is regular on R, so we get a short exact sequence

$$0 \longrightarrow R \xrightarrow{x} R \longrightarrow R/(x) \longrightarrow 0.$$

On the one hand, $\mathrm{H}_0(x; R) = R/(x)$, but

$$\mathrm{H}_0\left(x; R \xrightarrow{x} R\right) = R/(x) \xrightarrow{x} R/(x)$$

is the zero map. In particular, $\mathrm{H}_0(x; -)$ is not left exact.

More generally, $\mathrm{H}_0(\underline{x}; -)$ is naturally isomorphic to $- \otimes_R R/(\underline{x})$, which is typically not left exact. You might enjoy finding examples of the failure of $\mathrm{H}_i(\underline{x}; -)$ to be left or right exact for other values of i. However, $\mathrm{H}_0(\underline{x}; -) \cong R/(\underline{x}) \otimes_R -$ is in fact right exact.

Theorem 1.55. *If $x_1, \dots, x_n$ is a regular sequence on R, then* $\operatorname{height}(x_1, \dots, x_n) = n$. *Moreover, every minimal prime of $(x_1, \dots, x_n)$ has height n.*

Proof. By induction on n. When $n = 1$, x_1 is regular if and only if x_1 is not in the set of zero divisors of R. By Theorem 6.27 from Commutative Algebra I, this means x_1 is not in any associated prime of R, and in particular, x_1 is not in any of the minimal primes of R. Therefore, any prime containing x_1 must have height at least 1, so $\operatorname{height}(x_1) \geqslant 1$. By Krull's Height Theorem, Theorem 8.5 from Commutative Algebra I, we always have $\operatorname{height}(x_1) \leqslant 1$, so we conclude that $\operatorname{height}(x_1) = 1$.

Suppose that for some $n \geqslant 1$, all regular sequences of length n generate an ideal of height n. If $x_1, \dots, x_{n+1}$ is a regular sequence on R, then in particular x_{n+1} is regular on the quotient $R/(x_1, \dots, x_n)$. By the base case, the one-generated ideal $(x_1, \dots, x_{n+1})/(x_1, \dots, x_n)$ has height 1 in $R/(x_1, \dots, x_n)$. By induction hypothesis, $\operatorname{height}(x_1, \dots, x_n) = n$. We conclude that $\operatorname{height}(x_1, \dots, x_{n+1}) = n + 1$.

We have now shown that $\operatorname{height}(x_1, \dots, x_n) = n$, and thus every minimal prime of $x_1, \dots, x_n$ has height at least n. But Krulls Height Theorem says more: it tells us that every minimal prime of $(x_1, \dots, x_n)$ has height at most n. We conclude that every minimal prime of $x_1, \dots, x_n$ has height exactly n. □

The converse does not hold in general.

Example 1.56. Let $R = k[\![x, y]\!]/(x^2, xy)$. The element y is not in the unique minimal prime $(x)/(x^2, xy)$ of R, so $\operatorname{height}(y) = 1$. However, y is not regular.

We will later see that life is much more worth living over a Cohen-Macaulay ring, where indeed $\operatorname{height}(x_1, \dots, x_n) = n$ if and only if $x_1, \dots, x_n$ forms a regular sequence.

Remark 1.57. In the local/graded setting, another immediate consequence of Theorem 1.55 is that if $\underline{x} = x_1, \dots, x_n$ is a regular sequence, then $\underline{x}$ is a minimal generating set for $(\underline{x})$: if not, then $(\underline{x})$ would have less than n generators, and by Krull's Height Theorem (Theorem 8.5 from Commutative Algebra I) $\operatorname{height}(\underline{x}) < n$.

But in fact, more is true: if $\underline{x} = x_1, \dots, x_n$ is a regular sequence on any module M, then we claim that $\underline{x}$ form a minimal generating set for $(\underline{x})$. If not, then some x_i is a combination of the others; since the order does not matter, we can rename the elements and assume $x_n = a_1 x_1 + \cdots + a_{n-1} x_{n-1}$. Then x_n cannot be regular on $M/(x_1, \dots, x_{n-1})M$, as in fact it acts as zero on this quotient, and thus $\underline{x}$ is not a regular sequence on M.

1.6 Regular rings

Regular rings are the nicest possible kinds of rings, after fields.

Definition 1.58. The **embedding dimension** of a noetherian local ring $(R, \mathfrak{m}, k)$ is

$$\operatorname{embdim}(R) := \mu(\mathfrak{m}) = \dim_k \left(\mathfrak{m}/\mathfrak{m}^2\right).$$

To motivate the name embedding dimension, let us move to the setting of k algebras, and talk about the geometric reasons behind it. Let $Q = k[x_1, \dots, x_d]$ be a polynomial ring over the field k, and let I be a homogeneous ideal in Q. The geometry of the variety $V(I)$ is reflected in the algebra of the k-algebra $R = Q/I$. In particular, its dimension is the dimension of the ring R/I. Moreover, $V(I)$ is a subvariety of the affine space $\mathbb{A}_k^d$ of dimension d. If $I \subseteq (x_1, \dots, x_d)^2$, then one can show that we cannot embedded $V(I)$ in $\mathbb{A}_k^n$ for any $n < d$. Thus d is the *embedding dimension* of $V(I)$, and thus of R. Now note that d is the minimal number of generators of the unique homogeneous maximal ideal $(x_1, \dots, x_d)/I$ of R.

Exercise 11. Let $(R, \mathfrak{m})$ be a noetherian local ring and let I be an ideal in R. Show that

$$\operatorname{embdim}(R) = \operatorname{embdim}(R/I) + \dim_k \left(\frac{I + \mathfrak{m}^2}{\mathfrak{m}^2}\right).$$

Remark 1.59. Let $(R, \mathfrak{m})$ be a noetherian local ring. By Krull's Height Theorem,

$$\operatorname{embdim}(R) = \mu(\mathfrak{m}) \geqslant \operatorname{height}(\mathfrak{m}) = \dim(R).$$

A noetherian local ring is regular whenever $\mathfrak{m}$ has the smallest possible number of minimal generators, or equivalently, R has the smallest possible embedding dimension.

Definition 1.60. A noetherian local ring $(R, \mathfrak{m}, k)$ is a **regular local ring** if $\mathfrak{m}$ is minimally generated by $\dim(R)$ many elements, that is

$$\operatorname{embdim}(R) = \dim(R).$$

If $\dim(R) = d$ and $\mathfrak{m} = (x_1, \dots, x_d)$, the elements $\underline{x} = x_1, \dots, x_d$ are called a **regular system of parameters**. We might write RLR as shorthand for regular local ring.

Example 1.61. Any field is a regular local ring: it has dimension 0 and its maximal ideal is generated by no elements. Conversely, any artinian local ring is regular if and only if it is a field.

Example 1.62. Given any field k, the power series ring $k[\![x_1, \dots, x_d]\!]$ is a regular local ring: its dimension is d and $x_1, \dots, x_d$ form a regular system of parameters.

The localization of a polynomial ring at the homogeneous maximal ideal is also regular: $R = k[x_1, \dots, x_d]_{(x_1, \dots, x_d)}$ has dimension d and $x_1, \dots, x_d$ form a regular system of parameters.

Example 1.63. The localization $\mathbb{Z}_{(p)}$ of $\mathbb{Z}$ at the prime ideal (p) is a regular local ring: its dimension is height$(p) = 1$ and the maximal ideal is generated by p. This is an example of a DVR (discrete valuation ring), which is a topic we will not have a chance to explore in detail. However, one of the many equivalent definitions of DVR is that it is a regular local ring of dimension 1.

The localization $\mathbb{Z}[x]_{(p,x)}$ is also a regular local ring: the dimension of R is 2 and p, x form a regular system of parameters.

Example 1.64. The local ring $R = k[\![x, y]\!]/(x^2, xy)$ is not regular: it has dimension 1 but embedding dimension 2.

Exercise 12. Let $(R, \mathfrak{m})$ be a noetherian local ring. Show that if $\underline{x} = x_1, \dots, x_n$ is a regular sequence in R, then $\dim(R/(\underline{x})) = \dim(R) - n$.

Exercise 13. Let $(R, \mathfrak{m})$ be a regular local ring. Show that if $x \in \mathfrak{m}$ and $x \notin \mathfrak{m}^2$, then $R/(x)$ is a regular local ring of $\dim(R/(x)) = \dim(R) - 1$.

Example 1.65. For a slightly more exotic example, let p be a prime integer and $Q = \mathbb{Z}_{(p)}[x]_{(p,x)}$, and let $R = Q/(p - x^2)$. This is a regular local ring: $\dim(R) = 1 = \text{embdim}(R)$, as the maximal ideal is generated by x. Alternatively, we may note that $p - x^2 \in \mathfrak{m} \setminus \mathfrak{m}^2$, so Exercise 13 guarantees that R is a regular local ring.

Theorem 1.66. *Every regular local ring is a domain.*

Proof. We will do induction on $d = \dim(R)$. When $d = 0$, then $\mathfrak{m}$ must be generated by 0 elements, so $\mathfrak{m} = (0)$ and R is a field. In particular, R is a domain.

Now suppose that for some d, all regular local rings of dimension d are domains, and let R be a regular local rind of dimension $d + 1$. By the strong version of Prime Avoidance (see Lemma 3.29 from Commutative Algebra I), which says we can avoid one arbitrary ideal and any finite number of prime ideals, there exists

$$f \in \mathfrak{m} \setminus \mathfrak{m}^2 \bigcup_{P \in \text{Min}(R)} P.$$

Now by Exercise 13, $R/(f)$ is a regular local ring of dimension $d - 1$, since $f \notin \mathfrak{m}^2$. By induction hypothesis, $R/(f)$ must be a domain, so (f) is prime. Note that we chose f to not be in any minimal prime, so (f) has height 1. In particular, there is a minimal prime P of R such that

$$(0) \subseteq P \subsetneq (f).$$

Now given any $y \in P$, $y = rf$ for some $r \in R$, and since $rf = y \in P$ and $f \notin P$, we must have $r \in P$. Thus $y \in (f)P$, and since this holds for all $y \in P$, we conclude that $P = (f)P$. But $f \in \mathfrak{m}$, so by NAK (see Theorem 5.34 from Commutative Algebra I) we must have $P = (0)$. In particular, (0) is a prime ideal. □

Exercise 14. Let $(R, \mathfrak{m})$ be a regular local ring and let $x \in \mathfrak{m}$. Show that if $R/(x)$ is regular then $x \notin \mathfrak{m}^2$.

We are now ready to give a completely homological characterization of regular local rings. This characterization, first proved by Auslander and Buchsbaum and independently by Serre, solved a famous question called the Localization Problem.

Problem 1 (Localization Problem)**.** If R is a RLR, must R_P be a RLR for every prime P?

This question asks if being regular is a local property. A positive answer allows for the following global definition of regularity, as we will later see. But before we can get to this famous homological characterization of regular local rings, and the solution to the localization problem, we will need to sharpen our tools a bit.

Lemma 1.67. *Let $(R, \mathfrak{m}, k)$ be a noetherian local ring, and $F \xrightarrow{\varphi} G$ an R-module map between finitely generated free R-modules. If $\varphi \otimes_R k$ is injective, then $\varphi(F)$ if a free direct summand of G, and φ splits as a map of R-modules.*

Proof. Let $F = R^n$ with standard basis $\{e_1, \dots, e_n\}$, and let $G = R^m$. Our assumption that $\varphi \otimes_R k$ is injective implies that the images of $\varphi(e_1), \dots, \varphi(e_n)$ in $G/\mathfrak{m}G = k^m$ are linearly independent, so we can complete them to a basis $\overline{\varphi(e_1)}, \dots, \overline{\varphi(e_n)}, \overline{f_{n+1}}, \dots, \overline{f_m}$ for $G/\mathfrak{m}G$. By NAK (see Theorem 5.34 from Commutative Algebra I), we can lift those elements f_i to G so that $\varphi(e_1), \dots, \varphi(e_n), f_{n+1}, \dots, f_m$ is a basis for G. In particular, taking

$$H = \bigoplus_{i=n+1}^{m} Rf_i$$

the free module $\varphi(F) \cong F$ is a direct summand of G, via

$$G = \varphi(F) \oplus H.$$

Note that $\varphi(F)$ is a free module of rank n, and thus isomorphic to F. In particular, the map $\psi \colon G \to F$ given by projection onto $\varphi(F)$ is a splitting for φ. □

Theorem 1.68 (Serre)**.** *Let $(R, \mathfrak{m}, k)$ be a noetherian local ring. If $\mu(\mathfrak{m}) = s$, then*

$$\beta_i(k) \geqslant \binom{s}{i}.$$

Proof. Let $x_1, \dots, x_s$ be minimal generators for $\mathfrak{m}$, and set $K := \mathrm{kos}(x_1, \dots, x_s)$. If F is a minimal free resolution for $\mathfrak{m}$, we claim that K_i is a direct summand of F_i.

By Theorem 5.19 from Homological Algebra, the identity map on k lifts to a map of complexes $\varphi \colon K \longrightarrow F$. We claim that the maps φ_i split, which will prove our claim that K_i is a direct summand of F_i. The map $\varphi_0 \colon K_0 = R \longrightarrow F_0 = R$ must be the identity map, so φ_0 splits.

We proceed by induction on i. Suppose we have shown that φ_{i-1} splits, say by a splitting ψ_{i-1}. To be precise, this means that $\psi_{i-1}\varphi_{i-1} = \mathrm{id}_{F_{i-1}}$. By Lemma 1.67, it is enough to show that $\varphi_i \otimes_R k$ to be injective. To show $\varphi_i \otimes_R k$ is injective, we need to show that if

$z \in K_i$ is such that $\varphi_i(z) \in \mathfrak{m}F_i$, then $z \in \mathfrak{m}K_i$. First, we note that F is minimal and thus $\operatorname{im}\partial \subseteq \mathfrak{m}F$, by Theorem 5.9 from Homological Algebra, so

$$\partial_i\varphi_i(z) \in \mathfrak{m}^2F_i.$$

By commutativity, $\varphi_{i-1}\partial_i(z) \in \mathfrak{m}^2F_i$. Since $\psi_{i-1}\varphi_{i-1} = \mathrm{id}_{K_{i-1}}$, we must have

$$\partial_i(z) = \psi_{i-1}\varphi_{i-1}\partial_i(z) \in \mathfrak{m}^2K_{i-1}.$$

We claim that this implies $z \in \mathfrak{m}K_i$. To see that, we compute ∂_i explicitly: first we write z as a linear combination of our basis elements, say

$$z = \sum_{j_1<\cdots<j_i} z_j e_{j_1} \wedge \cdots \wedge e_{j_i}$$

so that

$$\partial_i(z) = \sum_{j,k} \pm z_j x_{j_k} e_{j_1} \wedge \cdots \wedge \widehat{e_{j_k}} \wedge \cdots \wedge e_{j_i}.$$

Now let us keep track of is the coefficients: rewriting this carefully in the basis for K_{i-1}, the coefficient for each basis element is a linear combination of terms of the form z_jx_l. We showed that $\partial_i(z) \in \mathfrak{m}^2F_{i-1}$, so each appropriate combination of z_jx_l is in $\mathfrak{m}^2$. Thus each such combination of z_jx_l is zero in $\mathfrak{m}/\mathfrak{m}^2$. We assumed that $x_1, \ldots, x_s$ were minimal generators for $\mathfrak{m}$ to begin with, and thus a basis for $\mathfrak{m}/\mathfrak{m}^2$, so that the images of our coefficients z_j must all be zero in $R/\mathfrak{m}$. Therefore, all our coefficients z_j must be in $\mathfrak{m}$. We conclude that indeed $z \in \mathfrak{m}F_i$, and thus that $\varphi_i \otimes_R k$ is injective. By Lemma 1.67, φ_i splits.

We have shown that K_i is a direct summand of F_i for all i, and thus

$$\beta_i(k) = \dim_k(\operatorname{Tor}_i^R(k,k)) = \dim_k(F_i \otimes_R k) \geqslant \dim_k(K_i \otimes_R k) = \binom{s}{i}. \quad \square$$

Theorem 1.68 does not hold if we replace k by another cyclic module R/I and $\mu(\mathfrak{m})$ by $\mu(I)$. However, there is a closely related conjectured inequality that remains open:

Conjecture 1.69 (Buchsbaum—Eisenbud, Horrocks)**.** *Let $(R, \mathfrak{m}, k)$ be a noetherian local ring of dimension d and M a finitely generated Artinian R-module with* $\operatorname{pdim}(M) < \infty$*. Then*

$$\beta_i(M) \geqslant \binom{d}{i}.$$

While this remains an open question, there is much evidence to support it. For example, the conjecture predicts the following inequality, previously know as the Total Rank Conjecture, which was recently shown by Walker in almost all cases, and by Walker and VandeBogert in characteristic 2.

Theorem 1.70 (Walker, 2017 [Wal17], VandeBogert–Walker, 2025 [VW25]). *Let $(R, \mathfrak{m}, k)$ be a noetherian local ring of dimension d and characteristic not 2, $M \neq 0$ a finitely generated R-module of finite projective dimension, and $c = \operatorname{height}(\operatorname{ann}(M))$. Then*

$$\sum_i \beta_i(M) \geqslant \sum_i \binom{c}{i} = 2^c.$$

We are finally ready for the famous homological characterization of regular rings that solved the Localization Problem:

Theorem 1.71 (Auslander–Buchsbaum, Serre [AB57, Ser56]). *Let $(R, \mathfrak{m}, k)$ be a noetherian local ring of dimension d. The following are equivalent:*

1) *The residue field k has finite projective dimension.*
2) *Every finitely generated R-module has finite projective dimension.*
3) *The maximal ideal $\mathfrak{m}$ is generated by a regular sequence.*
4) *The maximal ideal $\mathfrak{m}$ is generated by d elements.*

Proof. The implication 2 $\Rightarrow$ 1 is obvious, as $M = k$ is a finitely generated R-module. We proved 1 $\Rightarrow$ 2 in Corollary 1.24.

(3 $\Rightarrow$ 1) If $\mathfrak{m}$ is generated by a regular sequence $\underline{x}$, then the Koszul complex on $\underline{x}$ is a minimal free resolution of k, by Corollary 1.47. Therefore, k has projective dimension $\mu(\mathfrak{m})$. By Theorem 1.55, the length $\mu(\mathfrak{m})$ of that regular sequence is the height of $\mathfrak{m}$, which is d.

(1 $\Rightarrow$ 4) Suppose $\operatorname{pdim}(k) < \infty$. We claim that $\operatorname{pdim}(k) \leqslant \dim(R) = d$. By contradiction, suppose $\operatorname{pdim}(k) > d$ but $\operatorname{pdim}(k) < \infty$. Choose a maximal regular sequence $y_1, \dots, y_t \in \mathfrak{m}$. By Theorem 1.55, $t \leqslant d$. Moreover, every element in $\mathfrak{m}$ is a zerodivisor on $R/(y_1, \dots, y_t)$, or else we could add to our regular sequence. So $\mathfrak{m}$ is contained in the union of the zerodivisors on $R/(y_1, \dots, y_t)$, which by Theorem 6.27 from Commutative Algebra I is the same as the union of the associated primes of $R/(y_1, \dots, y_t)$. By Prime Avoidance (see Lemma 3.29 from Commutative Algebra I), $\mathfrak{m}$ must be contained in some associated prime of $R/(y_1, \dots, y_t)$. But $\mathfrak{m}$ is maximal, so $\mathfrak{m}$ *is* an associated prime of $R/(y_1, \dots, y_t)$. Equivalently, $k = R/\mathfrak{m}$ embeds into $R/(y_1, \dots, y_t)$. This gives us a short exact sequence

$$0 \longrightarrow k \longrightarrow R/(y_1, \dots, y_t) \longrightarrow M \longrightarrow 0.$$

We get a long exact sequence for Tor by Theorem 6.31 from Homological Algebra:

$$\cdots \longrightarrow \operatorname{Tor}_{i+1}^R(M, k) \longrightarrow \operatorname{Tor}_i^R(k, k) \longrightarrow \operatorname{Tor}_i^R(R/(y_1, \dots, y_t), k) \longrightarrow \cdots.$$

We know $t = \operatorname{pdim}(R/(y_1, \dots, y_t))$, by Corollary 1.47, so $\operatorname{Tor}_i^R(R/(y_1, \dots, y_t), k) = 0$ for all $i > t$. But $t \leqslant d < \operatorname{pdim}(k)$, so in particular $\operatorname{Tor}_i^R(R/(y_1, \dots, y_t), k) = 0$ for $i = \operatorname{pdim}(k)$.

Moreover, Corollary 1.24 says that $\operatorname{pdim}(M) \leqslant \operatorname{pdim}(k)$ for all finitely generated R-modules M, so in particular $\operatorname{Tor}_{i+1}^R(M,k) = 0$ for $i = \operatorname{pdim}(k)$. But this is impossible: our long exact sequence would then have $\operatorname{Tor}_{\operatorname{pdim}(k)}^R(k,k) \neq 0$ sandwiched between two zero modules.

Now suppose that $\operatorname{pdim}(k) < \infty$. We have seen that this implies that $\operatorname{pdim}(k) \leqslant \dim(R) = d$. By Theorem 1.68 and Exercise 2, for all i

$$\beta_i(k) = \dim_k(\operatorname{Tor}_i^R(k,k)) \geqslant \binom{\mu(\mathfrak{m})}{i}.$$

Since $\beta_i(k) = 0$ for all $i > \operatorname{pdim}(k)$, we must have

$$\mu(\mathfrak{m}) \leqslant \operatorname{pdim}(k) \leqslant \dim(R) = d.$$

But $\operatorname{height}(\mathfrak{m}) = \dim(R) = d$, so by Krull's Height theorem, Theorem 8.5 from Commutative Algebra I, $\mu(\mathfrak{m}) \geqslant d$. We conclude that $\mathfrak{m}$ is generated by exactly d elements, as desired.

(4 $\Rightarrow$ 3) Let $\mathfrak{m} = (x_1, \dots, x_d)$. If $d = 0$, then $\mathfrak{m} = (\{\}) = (0)$, and there is nothing to prove.

We proceed by induction on d, assuming that $d \geqslant 1$ and that we have shown that whenever $\mathfrak{m}$ is generated by $d-1$ elements $x_1, \dots, x_{d-1}$, then $x_1, \dots, x_{d-1}$ form a regular sequence.

Since there is at least one minimal generator x_1 and $x_1 \in \mathfrak{m} \setminus \mathfrak{m}^2$, by Exercise 13 we know that $R/(x_1)$ is a regular local ring of dimension $d-1$. Therefore, by induction hypothesis $x_2, \dots, x_d$ form a regular sequence on $R/(x_1)$. Moreover, by Theorem 1.66 R is a domain then x_1 is a regular element in R, so we conclude that $x_1, \dots, x_d$ form a regular sequence. □

Our proof also showed the following:

Corollary 1.72. *Every regular local ring $(R, \mathfrak{m}, k)$ has $\operatorname{pdim}(k) = \dim R$. Therefore, for all finitely generated R-modules M,*

$$\operatorname{pdim}(M) \leqslant \dim(R).$$

Now we can solve the Localization Problem.

Exercise 15. Show that if R is a regular local ring, then R_P is regular for every prime P.

This allows for the following global definition of regular ring:

Definition 1.73. A noetherian ring R is **regular** if R_P is a regular local ring for all prime ideals P.

Remark 1.74. If we want to show that a particular ring (not necessarily local) is regular, it is sufficient to show that $R_{\mathfrak{m}}$ is a regular local ring for every maximal ideal $\mathfrak{m}$: given any prime P, there is a maximal ideal $\mathfrak{m}$ containing P, and $R_P \cong (R_{\mathfrak{m}})_P$ is now a localization of the regular local ring $R_{\mathfrak{m}}$, which by Exercise 15 must be a regular local rings.

Remark 1.75. In a principal ideal domain R that is not a field, every maximal ideal is generated by exactly one element, so R is regular.

Hilbert's Syzygy Theorem (see Theorem 1.48) tells us that finitely generated *graded* modules over a polynomial ring $R = k[x_1, \dots, x_d]$ have finite projective dimension, but this is not quite enough to conclude that polynomial rings are regular. Nevertheless, they are indeed regular rings. This is elementary when k is algebraically closed:

Example 1.76. Let $R = k[x_1, \dots, x_d]$ with k an algebraically closed field. The maximal ideals in R are precisely those of the form $\mathfrak{m} = (x_1 - a_1, \dots, x_d - a_d)$, which are all generated by d elements, and they all have height d. Therefore, $R_{\mathfrak{m}}$ is a regular local ring for all maximal ideals $\mathfrak{m}$, and thus R is a regular ring.

The general case, however, requires a bit more work:

Theorem 1.77. *Every polynomial ring $R = k[x_1, \dots, x_d]$ over a field k is a regular ring.*

Proof. It is sufficient to show that $R_{\mathfrak{m}}$ is a regular local ring for every maximal ideal $\mathfrak{m}$. We are going to show that every maximal ideal is generated by d elements, which implies that $\mathfrak{m}_{\mathfrak{m}}$ is also generated by d elements. Since the height of every maximal ideal is d, by Theorem 7.46 from Commutative Algebra I, this will imply that $R_{\mathfrak{m}}$ is a regular local ring.

We will use induction on d to show that every maximal ideal is generated by exactly d elements. When $d = 1$, $k[x]$ is a principal ideal domain, and as noted in Remark 1.75, we are done.

When $d > 1$, we can do a change of variables such as in Lemma 7.40 from Commutative Algebra I so that $\mathfrak{m}$ has a minimal generator that is monic in x_d. Set $\mathfrak{n} := \mathfrak{m} \cap k[x_1, \dots, x_{d-1}]$. Then $k[x_1, \dots, x_{d-1}]/\mathfrak{n} \longrightarrow R/\mathfrak{m}$ is an integral extension, as x_d satisfies a monic polynomial with coefficients in $k[x_1, \dots, x_{d-1}]$. Now Theorem 1.44 from Commutative Algebra I says that in an integral extension of domains $A \subseteq B$, A is a field if and only if B is a field. Since $R/\mathfrak{m}$ is a field, then $L = k[x_1, \dots, x_{d-1}]/\mathfrak{n}$ is also a field. Therefore, $\mathfrak{n}$ is a maximal ideal in $k[x_1, \dots, x_{d-1}]$, so by induction hypothesis it must be generated by $d-1$ elements. Consider the map

$$\begin{aligned} R &\longrightarrow L[x_d] = \tfrac{k[x_1,\dots,x_{d-1}]}{\mathfrak{n}}[x_n] \cong R/\mathfrak{n}R \\ x_i &\longmapsto x_i. \end{aligned}$$

The image of $\mathfrak{m}$ in $L[x_d]$ is a maximal ideal, and $L[x_d]$ is a polynomial ring over a field in one variable, so by the case $d = 1$ the $\mathfrak{m}L[x_d] = \mathfrak{m}/\mathfrak{n}R$ must be generated by 1 element. Therefore, $\mathfrak{n}$ is generated by $d-1$ elements and $\mathfrak{m}/\mathfrak{n}R$ is generated by 1 element, so $\mathfrak{m}$ is generated by $d - 1 + 1 = d$ elements. □

In fact, this is a corollary of the following more general fact, which we will not prove:

Theorem 1.78. *If R is a regular ring, then the polynomial ring $R[x_1, \dots, x_n]$ is also regular.*

Our proof of Theorem 1.77 also showed that over any field k, every maximal ideal in $k[x_1, \dots, x_d]$ is generated by exactly d elements. This does not extend to all regular rings: while all maximal ideals in a regular ring of dimension d must be *locally* generated by at most d elements, *globally* they might need more than d generators.

Exercise 16. Let $R = \mathbb{Z}[\sqrt{-5}] \cong \mathbb{Z}[x]/(x^2+5)$. Show that R is a one-dimensional regular ring, but not all maximal ideals are principal.

Definition 1.79. Let R be a noetherian ring. The **global dimension** of R is

$$\operatorname{gldim}(R) := \sup\{\operatorname{pdim}_R(M) \mid M \text{ is a finitely generated } R\text{-module}\}.$$

In fact, one can show that if for some integer $n \geqslant 0$, all finitely generated R-modules have projective dimension at most n, then *every* R-module has projective dimension at most n. Therefore, the supremum in the definition of global dimension can be taken over all R-modules. Time permitting, we might prove this later.

We can now rewrite the Auslander–Buchsbaum–Serre theorem as follows:

Theorem 1.80. *Let $(R, \mathfrak{m}, k)$ be a noetherian local ring. Then $\operatorname{pdim}(k) < \infty$ if and only if $\operatorname{gldim}(R) < \infty$. Moreover, if $\operatorname{gldim}(R) < \infty$, then we must have $\operatorname{gldim}(R) = \dim(R)$.*

The global dimension matches the dimension more generally. For that, we need a lemma:

Lemma 1.81. *If R is a regular ring, then $\operatorname{gldim}(R) \leqslant \dim(R)$.*

Proof. There is nothing to prove when $\dim(R)$ is infinite, so assume $\dim(R) = d$ is finite. Let F be a free resolution for the finitely generated R-module M. Let P be any prime ideal of R, and

$$\Omega := \ker\left(F_{d-1} \xrightarrow{\partial_{d-1}} F_{d-2}\right).$$

Localization is exact, so $F \otimes_R R_P$ is a free resolution for M_P and Ω_P is the kernel of $\partial_{d-1} \otimes_R R_P$. By assumption, R_P is a regular local ring of dimension at most d, so by Theorem 1.71 we know that the projective dimension of M_P over R_P is at most d. Any resolution for M_P is a direct sum of a minimal resolution and a trivial complex, and a minimal resolution of M_P has length at most d. Thus Ω_P must be a free module over R_P.

This holds for any prime ideal P, so Ω is locally free. By Theorem 4.61 from Homological Algebra, Ω must be projective, and thus the complex

$$0 \longrightarrow \Omega \longrightarrow F_{d-1} \longrightarrow \cdots \longrightarrow F_0$$

is a projective resolution for M, of length d. We conclude that $\operatorname{pdim}_R(M) \leqslant d$. □

Theorem 1.82. *A noetherian ring R is regular of dimension d if and only if $\operatorname{gldim}(R) = d$.*

Proof. ($\Rightarrow$) If that R is a regular ring of dimension d, then by Lemma 1.81 we know that $\operatorname{gldim}(R) \leqslant d$. Let $\mathfrak{m}$ be a maximal ideal of height d. By assumption, $R_\mathfrak{m}$ is a regular local ring of dimension d. Since localization is exact, localizing a finite resolution for $R/\mathfrak{m}$ gives us a finite resolution for $(R/\mathfrak{m})_\mathfrak{m} \cong R_\mathfrak{m}/\mathfrak{m}_\mathfrak{m}$, which may however not be minimal. Therefore,

$$d \geqslant \operatorname{gldim}(R) \geqslant \operatorname{pdim}_R(R/\mathfrak{m}) \geqslant \operatorname{pdim}_{R_\mathfrak{m}}\left(\frac{R_\mathfrak{m}}{\mathfrak{m}_\mathfrak{m}}\right) = \dim\left(R_\mathfrak{m}\right) = d.$$

We conclude that $\operatorname{gldim}(R) = \dim(R)$.

($\Leftarrow$) Suppose R has finite global dimension d. Then for every maximal ideal $\mathfrak{m}$, $R/\mathfrak{m}$ has projective dimension at most d. Since localization is exact and finitely generated projectives over a noetherian ring are locally free, localizing a finite projective resolution for $R/\mathfrak{m}$ gives us a finite projective resolution for $(R/\mathfrak{m})_{\mathfrak{m}} \cong R_{\mathfrak{m}}/\mathfrak{m}_{\mathfrak{m}}$, and therefore $R_{\mathfrak{m}}$ is a regular local ring by Theorem 1.71. We have shown that $R_{\mathfrak{m}}$ is a RLR for all maximal ideals $\mathfrak{m}$, and thus R is a regular ring.

Our argument also shows that

$$d = \operatorname{gldim}(R) \geqslant \operatorname{pdim}_R(R/\mathfrak{m}) \geqslant \operatorname{pdim}_{R_{\mathfrak{m}}}\left(\frac{R_{\mathfrak{m}}}{\mathfrak{m}_{\mathfrak{m}}}\right) = \dim(R_{\mathfrak{m}}) = \operatorname{height}(\mathfrak{m}).$$

In particular, $\dim(R)$ is finite, and bounded above by d. By Lemma 1.81,

$$d = \operatorname{gldim}(R) \leqslant \dim(R).$$

We conclude that $\dim(R) = d$. □

Remark 1.83. Note that there are examples of noetherian regular rings with infinite Krull dimension, and their global dimension is thus also infinite. Nagata's Example 7.7 from Commutative Algebra I is one such ring.

However, if R is regular (even of infinite Krull dimension), then every finitely generated R-module has finite projective dimension; the issue is that there might be finitely generated R-modules of arbitrarily high projective dimension.

Macaulay2. Theorem 1.82 gives us a hint at how to decide if a module has infinite projective dimension: if its projective dimension is at least $d + 1$, then it must be infinite. This is particularly helpful for computations using Macaulay2: while the computer will only compute finitely many steps of the resolution, when looking at a graded R-module, if the computer returns a minimal resolution with at least $\dim(R) + 1$ steps, then we know that M must have infinite projective dimension.

We end this chapter with one more property of regular local rings: they are all UFDs.

Theorem 1.84 (Auslander–Buchsbaum, 1959)**.** *Any regular local ring is a UFD.*

However, not all regular rings are UFDs, only RLRs.

Example 1.85. The regular ring $R = \mathbb{Z}[\sqrt{-5}]$ from Exercise 16 is not a UFD:

$$6 = 2 \times 3 \quad \text{and} \quad 6 = (1 + \sqrt{-5})(1 - \sqrt{-5})$$

are two factorizations as products of irreducibles.

Exercises

Exercise 17. Let R be a noetherian local ring and M a finitely generated R-module with $\operatorname{pdim}(M) = t$. Show that for all nonzero finitely generated R-modules N,

$$\operatorname{Ext}^t_R(M, N) \neq 0.$$

Exercise 18. Let $(R, \mathfrak{m})$ be a noetherian local ring and A and B be R-modules such that A is a direct summand of B. Show that if $\operatorname{pdim}(B) < \infty$ then $\operatorname{pdim}(A) < \infty$.

Definition 1.86. Let k be a field and let $f_1, \dots, f_n$ be monomials in $k[x_1, \dots, x_d]$, minimally generating the ideal $I = (f_1, \dots, f_n)$. For each subset $J \subseteq [n] := \{1, \dots, n\}$, set

$$f_J = \operatorname{lcm}(f_j \mid j \in J).$$

The **Taylor resolution** of R/I is the complex (T, ∂) defined as follows:

- In homological degree s, T_s is the free R-module on basis e_J with

  $$J = \{j_1, \dots, j_s\} \subseteq [n]$$

  ranging over all the subsets of $[n]$ of size $|J| = s$.

- For each basis element e_J with $|J| = s$, the differential is defined as

  $$\partial(e_J) = \sum_{i=1}^{s} (-1)^{i+1} \frac{f_J}{f_{J \setminus \{j_i\}}} e_{J \setminus \{j_i\}}.$$

In the 1960s, Diana Taylor proved in her PhD thesis that this is a free resolution for I, which is now known as the **Taylor resolution**. We will use her theorem without proof.

Note: any monomial ideal has a unique minimal generating set consisting of monomials, so we can talk about the Taylor resolution of a monomial ideal I.

Exercise 19. Let k be a field and let $f_1, \dots, f_n$ be monomials in $k[x_1, \dots, x_d]$, minimally generating the monomial ideal $I = (f_1, \dots, f_n)$.

1) Find the Taylor resolution of $I = (xy, xz, yz) \subseteq k[x, y, z]$, clearly indicating each basis element.

2) Use the Taylor resolution of $I = (xy, xz, yz)$ to find the minimal resolution for I.

   Note that the goal of this problem is to understand how to minimize a nonminimal free resolution.

3) Let $I = (f_1, \dots, f_n)$ be a squarefree monomial ideal. Show that the Taylor resolution of I is minimal if and only if for each i there exists a variable y_i such that $y_i \mid f_i$ but $y_i \nmid f_j$ for all $j \neq i$.

Exercise 20. Let R be a ring, $I = (x_1, \dots, x_n)$ an ideal in R, and M a finitely generated R-module. Show that if $IM = M$, then $\operatorname{kos}(\underline{x}; M)$ is exact.

Exercise 21. Give an example showing that Koszul homology on $\underline{x}$ may depend on the choice of generators for $(\underline{x})$.

Exercise 22. Let M be a finitely generated R-module.

1) Show that if $\underline{x}$ is a regular sequence on M, then $\operatorname{Tor}_1^R(M, R/(\underline{x})) = 0$.

2) Show that if $\underline{x}$ is a regular sequence on M and also on R, then $\operatorname{Tor}_i^R(M, R/(\underline{x})) = 0$ for all $i \geqslant 1$.

3) Give an example of a ring R, a finitely generated R-module M, and a regular sequence $\underline{x}$ on M such that $\operatorname{Tor}_i^R(M, R/(\underline{x})) \neq 0$ for some $i \geqslant 2$.

Definition 1.87. Let R be a ring. The **support** of a complex of R-modules M is defined as

$$\operatorname{Supp}(M) := \{P \in \operatorname{Spec}(R) \mid M_P \text{ is not exact}\}.$$

Exercise 23. Let R be a noetherian ring and $I = (x_1, \dots, x_n)$ for some $\underline{x} = x_1, \dots, x_n \in R$.

1) Show that for all complexes M of R-modules,

$$\operatorname{Supp}(M) = \bigcup_i \operatorname{Supp}(\operatorname{H}_i(M)).$$

2) Show that $\operatorname{Supp}(\operatorname{kos}(\underline{x})) = \operatorname{Supp}(R/I)$.

Exercise 24. Let $(R, \mathfrak{m}, k)$ be a noetherian local ring and let F be a complex of finitely generated free R-modules, not necessarily bounded on either side.

1) Show that if f is a regular element on R, then F is exact if and only if $F \otimes_R R/(f)$ is exact.

2) Show that if $\underline{f}$ is a regular sequence, then F is exact if and only if $F \otimes_R R/(\underline{f})$ is exact.

3) Show that if R is regular, then F is exact if and only if $F \otimes_R R/\mathfrak{m}$ is exact.

Exercise 25. Let $(R, \mathfrak{m})$ be a noetherian local ring and I be a proper ideal in R.

1) Show that there is a short exact sequence

$$0 \longrightarrow \tfrac{I+\mathfrak{m}^2}{\mathfrak{m}^2} \longrightarrow \mathfrak{m}/\mathfrak{m}^2 \longrightarrow \tfrac{\mathfrak{m}}{\mathfrak{m}^2+I} \longrightarrow 0$$

and conclude that $\dim_k \left(\tfrac{I+\mathfrak{m}^2}{\mathfrak{m}^2}\right) = \operatorname{embdim}(R) - \operatorname{embdim}(R/I)$.

2) Show that if R and R/I are both regular rings, then there exists a minimal set of generators $x_1, \dots, x_d$ for $\mathfrak{m}$ such that $I = (x_1, \dots, x_n)$ for some n. In particular, I must be generated by a regular sequence.

3) If we only assume that R/I is a regular ring, must I be generated by a regular sequence?

Exercise 26. Let R be a noetherian ring and consider a regular element $f \in R$. Show that if R_f and $R/(f)$ are both regular rings, then R is a regular ring.

Exercise 27. Let k be any field. Determine if each of the following rings are regular, and carefully justify your answers.

1) $A = k[\![x, y, z]\!]/(x + y + z)$.

2) $B = k[\![x, y, z, w]\!]/(x^2, xy, yz, zw, w^2)$.

3) $C = k[\![x, y, z, w]\!]/(x, y) \cap (z, w)$.

4) $D = \mathbb{Z}[x, y]/(x^2 + y^2)$.

5) $E = \mathbb{Z}[x]/(x^2 - 1)$.

Chapter 2

Cohen-Macaulay rings

2.1 Depth

We now get back to regular sequences to talk about how long they can be.

Definition 2.1. Let I be an ideal in a noetherian ring R and M be an R-module. The **I-depth** of M is the maximal length of a regular sequence on M consisting of elements in I, denoted $\operatorname{depth}_I(M)$. The **grade** of an ideal I, denoted $\operatorname{grade}(I)$, is the maximal length of a regular sequence inside I, meaning that

$$\operatorname{grade}(I) = \operatorname{depth}_I(R).$$

When $(R, \mathfrak{m})$ is a local ring, the **depth** of M is $\operatorname{depth}(M) = \operatorname{depth}_{\mathfrak{m}}(M)$.

Remark 2.2. In a noetherian local ring $(R, \mathfrak{m})$, $\operatorname{depth}(R) = \operatorname{grade}(\mathfrak{m})$.

A priori, two maximal regular sequences on M inside I may have different lengths; we will soon see that is not the case. Before we prove that, we already have an upper bound for depth.

Remark 2.3. If $x_1, \dots, x_n$ is a regular sequence on R inside I, we saw in Theorem 1.55 that $\operatorname{height}(x_1, \dots, x_n) = n$, so $\operatorname{grade}(I) \leqslant \operatorname{height}(I)$. In particular, $\operatorname{depth}(R) \leqslant \dim(R)$.

Example 2.4. Let k be any field and $R = k[\![x_1, \dots, x_d]\!]$. The variables $x_1, \dots, x_d$ form a regular sequence, so $\operatorname{depth}(R) \geqslant d$. On the other hand, $\operatorname{depth}(R) \leqslant \dim(R) = d$, so $\operatorname{depth}(R) = d$.

Lemma 2.5. *Let R be a noetherian ring, I an ideal in R, and $M \neq 0$ a finitely generated R-module such that $IM \neq M$. There exists $r \in I$ which is regular on M if and only if $I \nsubseteq P$ for all $P \in \operatorname{Ass}(M)$. If $(R, \mathfrak{m})$ is local/graded, and M is a graded module in the graded setting, $\mathfrak{m} \in \operatorname{Ass}(M)$ if and only if there are no regular elements on M.*

Proof. If $r \in I$ is regular on M, then r is not in the union of the associated primes of M, by Theorem 6.27 from Commutative Algebra I. As a consequence, I cannot be contained in any associated prime of M.

Conversely, recall that M has finitely many associated primes, by Corollary 6.35 from Commutative Algebra I. If I is not contained in any associated prime of M, then by Prime Avoidance, Lemma 3.29 from Commutative Algebra I, it also cannot be contained in the union of the associated primes of M. Recall Theorem 6.27 from Commutative Algebra I, which says that the union of the associated primes is the set of zero divisors. We conclude that I contains some regular element on M.

In the graded case, recall that all associated primes of a graded module must be homogeneous, and thus they are all contained in the homogeneous maximal ideal $\mathfrak{m}$. Thus in the local/graded case, $\mathfrak{m} \in \operatorname{Ass}(M)$ if and only if $\mathfrak{m}$ is contained in some associated prime of M. We conclude that $\mathfrak{m} \in \operatorname{Ass}(M)$ if and only if there are no regular elements on M. □

We can construct maximal regular sequences explicitly.

Construction 2.6. Let R be a noetherian ring, I an ideal in R, and $M \neq 0$ a finitely generated R-module with $IM \neq M$. To construct a regular sequence on M inside I, we start by finding a regular element on M inside I. Either I is contained in some associated prime of M, in which case every element in I is a zerodivisor on M, or there exists an element $x_1 \in I$ not in any associated prime of M, by Prime Avoidance, Lemma 3.29 from Commutative Algebra I. In the graded case, homogeneous Prime Avoidance guarantees we can find such x_1 that is homogeneous. Such an element x_1 is regular on M, since the union of the associated primes of M is precisely the set of zerodivisors, by Theorem 6.27 from Commutative Algebra I.

Now we repeat the process: either I is contained in some associated prime of $M/(x_1)M$, in which case there are no regular elements on $M/(x_1)M$ inside I, or we can find $x_2 \in I$ not in any associated prime of $M/(x_1)M$, so x_2 is necessarily regular on $M/(x_1)M$. At each step, $(x_1, \dots, x_i)M \subsetneq (x_1, \dots, x_{i+1})M$, and since M is noetherian, the process must stop. In the graded setting, we have in particular proved we can construct a maximal regular sequence that consists entirely of homogeneous elements.

In fact, any regular sequence on M inside of I leads to such an increasing sequence of submodules of M, so all such sequences are finite.

Lemma 2.7. *Let R be a noetherian ring and M and N be finitely generated R-modules. If $a \in \operatorname{ann}_R(M)$ and $b \in \operatorname{ann}_R(N)$, then $a \operatorname{Ext}^i_R(M,N) = 0$ and $b \operatorname{Ext}^i_R(M,N) = 0$ for all i.*

Proof. When $i = 0$, we want to show that $a \in \operatorname{ann}(\operatorname{Hom}_R(M,N))$ and $b \in \operatorname{ann}(\operatorname{Hom}_R(M,N))$. Given any $f \in \operatorname{Hom}_R(M,N)$ and any $m \in M$, we know $am = 0$, so

$$af(m) = f(am) = f(0) = 0.$$

We conclude that $af = 0$. Moreover, b kills every element in N, so $bf = 0$. Now let $P \longrightarrow M$ be a projective resolution on M, and $N \longrightarrow E$ be an injective resolution of N. Then

$$\begin{aligned}\operatorname{Ext}^i_R(M,N) &= \mathrm{H}^i\left(0 \longrightarrow \operatorname{Hom}_R(P_0,N) \longrightarrow \operatorname{Hom}_R(P_1,N) \longrightarrow \operatorname{Hom}_R(P_2,N) \longrightarrow \cdots \right) \\ &= \mathrm{H}^i\left(0 \longrightarrow \operatorname{Hom}_R(M,E^0) \longrightarrow \operatorname{Hom}_R(M,E^1) \longrightarrow \operatorname{Hom}_R(M,E^2) \longrightarrow \cdots \right),\end{aligned}$$

so $\operatorname{Ext}^i_R(M,N)$ is a subquotient of both $\operatorname{Hom}_R(P_i,N)$ and $\operatorname{Hom}_R(M,E^i)$. We conclude that a and b both kill $\operatorname{Ext}^i_R(M,N)$. □

Theorem 2.8. *Let R be a noetherian ring, I be an ideal in R, and M be a finitely generated R-module with $M \neq IM$. All maximal regular sequences on M inside I have the same length.*

Proof. Let $x_1, \dots, x_n \in I$ and $y_1, \dots, y_\ell \in I$ both be maximal regular sequences on M, and assume $n \leqslant \ell$.

When $n = 0$, every element in I is a zerodivisor on M, so $\ell = 0$.

When $n = 1$, we cannot extend the regular sequence x_1 further, so every element in I is a zerodivisor on $M/(x_1)M$. By Lemma 2.5, $I \subseteq P$ for some $P \in \operatorname{Ass}(M/(x_1)M)$. Thus there exists some $m \in M$, $m \notin (x_1)M$, such that $I \subseteq ((x_1)M :_R m)$, so $Im \subseteq (x_1)M$.

In particular, since $y_1 \in I$, we have $y_1 m = x_1 a$ for some $a \in M$. If $a \in (y_1)M$, then we would have

$$y_1 m = x_1 a \in (x_1 y_1)M.$$

Thus for some $n \in M$, we have

$$y_1 m = y_1 x_1 n \implies y_1(m - x_1 n) = 0.$$

Since y_1 is regular on M, that would imply $m \in (x_1)M$, which is a contradiction. We conclude that $a \notin (y_1)M$. Moreover,

$$(x_1)I \cdot a = I(x_1 a) = I(y_1 m) = y_1(Im) \subseteq (y_1)(x_1)M = (x_1)(y_1)M.$$

Since x_1 is a regular element on M, we must have $Ia \subseteq (y_1)M$. Therefore, $a \in M$ is an element that both satisfies $a \notin (y_1)M$ and $Ia \subseteq (y_1)M$, so every element in I kills a in $M/(y_1)M$. Therefore, every element in I is a zerodivisor on $M/(y_1)M$. This proves we cannot extend the regular sequence y_1, and thus $\ell = 1$.

We proceed by induction on n. Now assume that $n > 1$ and $\ell > n$. In particular, I contains a regular element on $M/(x_1, \dots, x_i)$ for all $i < n$ and a regular element on $M/(y_1, \dots, y_j)$ for all $j < \ell$, so by Prime Avoidance[1] we can pick $c \in I$ that avoids both all the (finitely many) associated primes of $M/(x_1, \dots, x_i)M$ for all $i < n$ and $M/(y_1, \dots, y_j)M$ for all $j < \ell$. In particular, $x_1, \dots, x_{n-1}, c$ and $y_1, \dots, y_n, c$ are both regular sequences on M. Now x_n and c are both regular sequences on $M/(x_1, \dots, x_{n-1})$, so the case $n = 1$ says $x_1, \dots, x_{n-1}, c$ is also a maximal regular sequence on M. Now by Lemma 1.51, $x_1, \dots, c, x_{n-1}$ is also a regular sequence on M, since c is also regular on $M/(x_1, \dots, x_{n-2})$, and so on, until we conclude that $c, x_1, \dots, x_{n-1}$ is a regular sequence on M. Similarly, $c, y_1, \dots, y_n$ is a regular sequence on M. Notice in fact that $c, x_1, \dots, x_{n-1}$ is maximal inside I, or else we could increase its size, move c back to after x_{n-1}, and obtain a contradiction. Therefore, $x_1, \dots, x_{n-1}$ and $c, y_1, \dots, y_n$ are both regular sequences on $M/(c)M$, and $x_1, \dots, x_{n-1}$ is maximal. But by induction hypothesis, all maximal regular sequences on $M/(c)M$ inside I have the same length, which would say that the length of $y_1, \dots, y_n$, n, is at most $n - 1$. This is a contradiction, so we conclude that $\ell = n$. □

[1] See Lemma 3.29 from Commutative Algebra I.

Remark 2.9. Theorem 2.8 tells us that $\operatorname{depth}_I(M)$ is not just the largest length of a regular sequence on M inside I: it is in fact the length of *any* maximal regular sequence on M inside I, where maximal simply means we cannot extend it to be any longer.

We will now show that depth can be described in a purely homological way.

Theorem 2.10. *Let R be a noetherian ring and M a finitely generated R-module. Then*

$$\operatorname{depth}_I(M) = \min\{i \mid \operatorname{Ext}^i_R(R/I, M) \neq 0\}.$$

Proof. When $\operatorname{depth}_I(M) = 0$, there is no regular sequence on M inside I. By Lemma 2.5, $I \subseteq P$ for some $P \in \operatorname{Ass}(M)$. We have an inclusion $R/P \hookrightarrow M$, so consider the composition

$$R/I \twoheadrightarrow R/P \hookrightarrow M.$$

This is a nonzero homomorphism of R-modules, so $\operatorname{Ext}^0(R/I, M) = \operatorname{Hom}(R/I, M) \neq 0$. On the other hand, if $\operatorname{Hom}_R(R/I, M) = \operatorname{Ext}^0(R/I, M) \neq 0$, then there exists a nonzero R-module homomorphism $R/I \longrightarrow M$. But to choose an R-module homomorphism $R/I \longrightarrow M$ is the same as choosing an element in M that is killed by I, so I contains no nonzero divisors on M and $\operatorname{depth}_I(M) = 0$.

We proceed by induction on $\operatorname{depth}_I(M) = n$, assuming the statement holds whenever $\operatorname{depth}_I(M) < n$. Let $x_1, \dots, x_n$ be a maximal regular sequence on M inside I. Then $x_2, \dots, x_n$ is a maximal regular sequence on $M/(x_1)M$, so by induction we know

$$n - 1 = \min\{i \mid \operatorname{Ext}^i_R(R/I, M) \neq 0\}.$$

Applying $\operatorname{Hom}_R(R/I, -)$ to the short exact sequence

$$0 \longrightarrow M \xrightarrow{\cdot x_1} M \longrightarrow M/(x_1)M \longrightarrow 0$$

we get a long exact sequence

$$\cdots \longrightarrow \operatorname{Ext}^{i-1}_R(R/I, M/(x_1)M) \longrightarrow \operatorname{Ext}^i_R(R/I, M) \xrightarrow{x_1} \operatorname{Ext}^i_R(R/I, M) \longrightarrow \cdots.$$

We know $\operatorname{Ext}^{n-1}_R(R/I, M/(x_1)M) \neq 0$ and $\operatorname{Ext}^i_R(R/I, M/(x_1)M) = 0$ for $i < n-1$. Therefore, whenever $i < n-1$,

$$\operatorname{Ext}^i_R(R/I, M) \xrightarrow{x_1} \operatorname{Ext}^i_R(R/I, M)$$

is an isomorphism. However, $x_1 \in I = \operatorname{ann}(R/I)$, so $\operatorname{ann}(R/I) \subseteq \operatorname{ann}(\operatorname{Ext}^i_R(R/I, M))$ by Lemma 2.7. Therefore, $\operatorname{Ext}^i_R(R/I, M) = 0$ for all $i < n-1$. Moreover, multiplication by x_1 is the zero map on $\operatorname{Ext}^i_R(R/I, M)$ for any i, also by Lemma 2.7. Finally, we have an exact sequence

$$\cdots \longrightarrow \operatorname{Ext}^{n-1}_R(R/I, M) \xrightarrow{x_1} \operatorname{Ext}^{n-1}_R(R/I, M) \longrightarrow \operatorname{Ext}^{n-1}_R(R/I, M/(x_1)M) \longrightarrow \cdots.$$

where the multiplication by x_1 maps are 0, so our exact sequence is

$$0 \to \operatorname{Ext}_R^{n-1}(R/I, M) \xrightarrow{0} \operatorname{Ext}_R^{n-1}(R/I, M) \to \underbrace{\operatorname{Ext}_R^{n-1}(R/I, M/x_1M)}_{\neq 0} \to \operatorname{Ext}_R^{n}(R/I, M) \xrightarrow{0} \cdots.$$

In particular, $\operatorname{Ext}^{n-1}(R/I, M) = 0$ and $\operatorname{Ext}^{n}(R/I, M) \neq 0$. We conclude that

$$\operatorname{depth}_I(M) = n = \min\{i \mid \operatorname{Ext}_R^i(R/I, M) \neq 0\}. \quad \square$$

For yet another homological characterization of depth, we turn to Koszul homology.

Theorem 2.11 (Depth sensitivity of the Koszul complex)**.** *Let R be a noetherian ring and M be finitely generated R-module. Given any $\underline{x} = x_1, \dots, x_n$ such that $(\underline{x})M \neq M$,*

$$\operatorname{depth}_{(\underline{x})}(M) = \max\{r \mid \mathrm{H}_i(\underline{x}; M) = 0 \text{ for all } i > n - r\}.$$

So we can measure $\operatorname{depth}_I(M)$ by looking at the first nonzero Koszul homology we see when we start counting *from the top*.

$$\begin{array}{lccccccccc} \operatorname{kos}(\underline{x}; M) & 0 \to & M & \to \cdots \to & M^{\binom{n}{n-r+1}} & \to & M^{\binom{n}{n-r}} & \to \cdots \to & M & \to 0 \\ \mathrm{H}(\underline{x}; M) & & 0 & \cdots & 0 & & \neq 0. & & & \end{array}$$

Proof. We are going to show that if $I = (\underline{x})$ contains elements $\underline{y} = y_1, \dots, y_m$ such that $\underline{y}$ is a regular sequence on M, then

$$\mathrm{H}_{n-i+1}(\underline{x}; M) = 0 \text{ for all } i = 1, \dots, m \text{ and } \mathrm{H}_{n-m}(\underline{x}; M) \cong \operatorname{Ext}_R^m(R/I, M).$$

This will prove the theorem: depth is the largest possible m we could take, and Theorem 2.10 says $\operatorname{Ext}_R^{\operatorname{depth}}(R/I, M) \neq 0$.

We proceed by induction on m. When $m = 0$, Theorem 1.35 says that

$$\mathrm{H}_n(\underline{x}; M) \cong (0 :_M I) \cong \operatorname{Hom}_R(R/I, M) \cong \operatorname{Ext}_R^m(R/I, M),$$

and we are done. When $m > 0$, the short exact sequence

$$0 \longrightarrow M \xrightarrow{y_1} M \longrightarrow M/(y_1)M \longrightarrow 0$$

induces a long exact sequence in koszul homology (see Theorem 1.35)

$$\cdots \longrightarrow \mathrm{H}_{i+1}(\underline{x}; M/(y_1)M) \longrightarrow \mathrm{H}_i(\underline{x}; M) \xrightarrow{y_1} \mathrm{H}_i(\underline{x}; M) \longrightarrow \mathrm{H}_i(\underline{x}; M/(y_1)M) \longrightarrow \cdots.$$

Now by Theorem 1.35, I kills $\mathrm{H}_i(\underline{x}; M)$, so the multiplication by y_1 map is zero in the long exact sequence above, which must then break into short exact sequences

$$0 \longrightarrow \mathrm{H}_i(\underline{x}; M) \longrightarrow \mathrm{H}_i(\underline{x}; M/(y_1)M) \longrightarrow \mathrm{H}_{i-1}(\underline{x}; M) \longrightarrow 0.$$

Since $y_2, \dots, y_m$ form a regular sequence of $m-1$ elements on $M/(y_1)M$ inside $(\underline{x})$, so by induction hypothesis

$$\mathrm{H}_{n-i+1}(\underline{x}; M/(y_1)M) = 0 \text{ for all } i = 1, \dots, m-1$$

and

$$\mathrm{H}_{n-m+1}(\underline{x}; M/(y_1)M) \cong \mathrm{Ext}_R^{m-1}(R/I, M/(y_1)M).$$

Therefore, for all $i \leqslant m-1$, we have an exact sequence

$$0 \longrightarrow \mathrm{H}_{n-i+1}(\underline{x}; M) \longrightarrow \underbrace{\mathrm{H}_{n-i+1}(\underline{x}; M/(y_1)M)}_{0} \longrightarrow \mathrm{H}_{n-i}(\underline{x}; M) \longrightarrow 0.$$

This implies the other two terms are zero also. The left hand side gives us $\mathrm{H}_{n-i+1}(\underline{x}; M) = 0$ for all $i = 1, \dots, m-1$. The right hand side gives us $\mathrm{H}_{n-m+1}(\underline{x}; M) = \mathrm{H}_{n-(m-1)}(\underline{x}; M) = 0$. Therefore, $\mathrm{H}_{n-i+1}(\underline{x}; M) = 0$ for all $i = 1, \dots, m$.

Moreover, we have a short exact sequence

$$0 \longrightarrow \underbrace{\mathrm{H}_{n-m+1}(\underline{x}; M)}_{0} \longrightarrow \mathrm{H}_{n-m+1}(\underline{x}; M/(y_1)M) \longrightarrow \mathrm{H}_{n-m}(\underline{x}; M) \longrightarrow 0.$$

so by induction hypothesis

$$\mathrm{H}_{n-m}(\underline{x}; M) \cong \mathrm{H}_{n-m+1}(\underline{x}; M/(y_1)M) \cong \mathrm{Ext}_R^{m-1}(R/I, M/(y_1)M).$$

Finally, we claim that

$$\mathrm{Ext}_R^m(R/I, M) \cong \mathrm{Ext}_R^{m-1}(R/I, M/(y_1)M).$$

To show that, consider the short exact sequence

$$0 \longrightarrow M \xrightarrow{y_1} M \longrightarrow M/(y_1)M \longrightarrow 0,$$

and the long exact sequence induced by applying $\mathrm{Ext}_R^i(R/I, -)$:

$$\cdots \longrightarrow \mathrm{Ext}_R^i(R/I, M) \xrightarrow{y_1} \mathrm{Ext}_R^i(R/I, M) \longrightarrow \mathrm{Ext}_R^i(R/I, M/(y_1)M) \longrightarrow \cdots.$$

We assumed we have a regular sequence on M of length m inside I, so $\mathrm{depth}_I(M) \geqslant m$. Therefore, $\mathrm{Ext}_R^{m-1}(R/I, M) = 0$ by Theorem 2.10. Since $y_1 \in I$, by Lemma 2.7 multiplication by y_1 on $\mathrm{Ext}_R^i(R/I, M)$ is the zero map, so we get an exact sequence

$$0 \longrightarrow \mathrm{Ext}_R^{m-1}(R/I, M/(y_1)M) \longrightarrow \mathrm{Ext}_R^m(R/I, M) \longrightarrow 0.$$

Therefore,

$$\mathrm{Ext}_R^m(R/I, M) \cong \mathrm{Ext}_R^{m-1}(R/I, M/(y_1)M),$$

which finishes our proof. □

Remark 2.12. If $I = (x_1, \dots, x_n)$ and $\operatorname{depth}_I(M) = n$, then by Theorem 2.11 the Koszul complex $\operatorname{kos}(\underline{x}; M)$ must be exact. This does not necessarily say that $\underline{x}$ is a regular sequence, only that there exists some regular sequence on M of length n inside I. However, that implication does hold in the local or graded setting, by Theorem 1.49.

Another useful property of depth is how it behaves across regular sequences; we leave the proof for the next problem set.

Theorem 2.13 (The Depth Lemma)**.** *Let R be a noetherian local ring and consider a short exact sequence of nonzero finitely generated R-modules*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0.$$

Then the following hold:

$$\begin{aligned}\operatorname{depth}(B) &\geqslant \min\{\operatorname{depth}(A), \operatorname{depth}(C)\}\\ \operatorname{depth}(A) &\geqslant \min\{\operatorname{depth}(B), \operatorname{depth}(C)+1\}\\ \operatorname{depth}(C) &\geqslant \min\{\operatorname{depth}(B), \operatorname{depth}(A)-1\}.\end{aligned}$$

Lemma 2.14. *Let $(R, \mathfrak{m}, k)$ be a noetherian local ring and $M \neq 0$ a finitely generated R-module of finite projective dimension. If $x \in R$ is regular on both R and M, then $M/(x)M$ has finite projective dimension over $R/(x)$, given by*

$$\operatorname{pdim}_{R/(x)}(M/(x)) = \operatorname{pdim}_R(M).$$

Proof. Since x is regular on R, the quotient $R/(x)$ is resolved by the Koszul complex, by Corollary 1.47, so $\operatorname{pdim}(R/(x)) = 1$. Therefore, $\operatorname{Tor}_i^R(R/(x), M) = 0$ for all $i \geqslant 2$. Let

$$0 \longrightarrow R^{\beta_n} \xrightarrow{\varphi_n} R^{\beta_{n-1}} \xrightarrow{\varphi_{n-1}} \cdots \longrightarrow R^{\beta_1} \xrightarrow{\varphi_1} R^{\beta_0} \xrightarrow{\varphi_0} M \longrightarrow 0$$

be a minimal free resolution for M. In particular, if we choose basis for each R^{β_i}, the entries in the matrices representing φ_i are all in $\mathfrak{m}$. Applying $- \otimes_R R/(x)$, we get a complex

$$0 \longrightarrow (R/(x))^{\beta_n} \longrightarrow (R/(x))^{\beta_{n-1}} \longrightarrow \cdots \longrightarrow (R/(x))^{\beta_0} \longrightarrow M/xM \longrightarrow 0$$

which is exact at $M \otimes_R R/(x) \cong M/xM$, since tensor is right exact, and whose homology is otherwise given by $\operatorname{Tor}_i^R(R/(x), M)$. In particular, our complex is exact for all $i \geqslant 2$, since we have seen that $\operatorname{Tor}_i^R(R/(x), M) = 0$ for all $i \geqslant 2$. The only remaining homology with a chance of being interesting is given by

$$\operatorname{Tor}_1^R(R/(x), M) = \operatorname{H}_1(M \otimes (0 \to R \xrightarrow{x} R \to 0)) = \operatorname{H}_1(0 \to M \xrightarrow{x} M \to 0) = (0 :_M x).$$

By assumption, x is regular on M, so $\operatorname{Tor}_1^R(R/(x), M) = (0 :_M x) = 0$. So the complex above is exact, and thus a free resolution for M/xM over $R/(x)$. In fact, the maps in this free resolution for $M/(x)M$ were obtained by tensoring φ with $R/(x)$, so we can obtain matrices representing each map by taking the matrix representing φ_i and setting all the entries in (x) equal to 0. In particular, all the entries are still in $\mathfrak{m}/(x)$, and our resolution for M/xM over $R/(x)$ is minimal. We conclude that

$$\operatorname{pdim}_{R/(x)}(M/xM) = \operatorname{pdim}_R(M). \qquad \square$$

We now have all the tools we need to prove a very useful formula relating depth and projective dimension.

Theorem 2.15 (Auslander–Buchsbaum Formula)**.** *Let $(R, \mathfrak{m}, k)$ be a noetherian local ring and $M \neq 0$ a finitely generated R-module of finite projective dimension. Then*

$$\operatorname{depth}(M) + \operatorname{pdim}_R(M) = \operatorname{depth}(R).$$

Proof. Suppose $\operatorname{depth}(R) = 0$. In that case, the claim is that $\operatorname{pdim}(M) = 0 = \operatorname{depth}(M)$. First, note that the fact that $\operatorname{depth}(R) = 0$ implies immediately that $\mathfrak{m} \in \operatorname{Ass}(R)$, by Lemma 2.5, so $\mathfrak{m}$ kills some nonzero $r \in R$. Consider a minimal free resolution for M, say

$$0 \longrightarrow F_n \xrightarrow{\varphi_n} F_{n-1} \xrightarrow{\varphi_{n-1}} \cdots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} M \longrightarrow 0.$$

Suppose $n > 0$, so that $\varphi_n \neq 0$. By minimality, $\varphi_n(F_n) \subseteq \mathfrak{m}F_{n-1}$, so

$$\varphi_n(r, 0, \cdots, 0) = r\varphi_n(1, 0, \cdots, 0) \in r\mathfrak{m} = 0$$

and φ_n is not injective. This is a contradiction, so we must have $n = 0$, and M is free, say $M \cong R^a$. Therefore, $\operatorname{pdim}_R(M) = 0$ and $\operatorname{depth}(M) = \operatorname{depth}(R^a) = \operatorname{depth}(R) = 0$. This proves the formula holds whenever $\operatorname{depth}(R) = 0$.

Now assume that $\operatorname{depth}(M) = 0$. Set $n = \operatorname{pdim}(M)$ and $t = \operatorname{depth}(R)$, and fix a maximal regular sequence $x_1, \dots, x_t \in \mathfrak{m}$. By Corollary 1.47, $\operatorname{pdim}(R/(x_1, \dots, x_t)) = t$. Our goal is to show that $n = t$. Notice that $\operatorname{Tor}_i^R(R/(x_1, \dots, x_t), M)$ can be computed via minimal free resolutions for either M or $R/(x_1, \dots, x_t)$, so $\operatorname{Tor}_i^R(R/(x_1, \dots, x_t), M)$ vanishes for $i > \min\{t, n\}$. We are going to show that both $\operatorname{Tor}_t^R(R/(x_1, \dots, x_t), M) \neq 0$ and $\operatorname{Tor}_n^R(R/(x_1, \dots, x_t), M) \neq 0$, which proves that $n = t$.

By Corollary 1.47, the Koszul complex is a minimal free resolution for $R/(x_1, \dots, x_t)$, so the last map in the minimal free resolution looks like

$$0 \longrightarrow R \xrightarrow{\begin{pmatrix} x_1 & -x_2 & x_3 & \cdots & (-1)^{t+1}x_t \end{pmatrix}^T} R^t.$$

Applying $- \otimes_R M$ gives

$$0 \longrightarrow M \xrightarrow{\begin{pmatrix} x_1 & -x_2 & x_3 & \cdots & (-1)^{t+1}x_t \end{pmatrix}^T} M^t.$$

Therefore,

$$\operatorname{Tor}_t^R(R/(x_1, \dots, x_t), M) = \bigcap_{i=1}^{t} \ker(M \xrightarrow{\pm x_i} M).$$

Our assumption that $\operatorname{depth}(M) = 0$ says that there are no regular elements on M, so by Lemma 2.5, we must have $\mathfrak{m} \in \operatorname{Ass}(M)$. Therefore, there exists a nonzero element $m \in M$ such that $\operatorname{ann}(m) = \mathfrak{m} \supseteq (x_1, \dots, x_t)$, so

$$m \in \bigcap_{i=1}^{t} \ker(M \xrightarrow{\pm x_i} M) = \operatorname{Tor}_t^R(R/(x_1, \dots, x_t), M)$$

and $\operatorname{Tor}_t^R(R/(x_1,\dots,x_t),M) \neq 0$.

On the other hand, to compute $\operatorname{Tor}_n^R(R/(x_1,\dots,x_t),M)$ we can take a minimal free resolution of M, say

$$0 \longrightarrow R^{\beta_n} \xrightarrow{\varphi_n} R^{\beta_{n-1}} \xrightarrow{\varphi_{n-1}} \cdots \longrightarrow R^{\beta_1} \xrightarrow{\varphi_1} R^{\beta_0} \xrightarrow{\varphi_0} M \longrightarrow 0,$$

and apply $-\otimes_R R/(\underline{x})$, so that $\operatorname{Tor}_n^R(R/(x_1,\dots,x_t),M)$ is the kernel of

$$(R/(x_1,\dots,x_t))^{\beta_n} \longrightarrow (R/(x_1,\dots,x_t))^{\beta_{n-1}}.$$

Our assumption that $x_1,\dots,x_t$ is a maximal regular sequence on R implies that any other element in R is a zerodivisor on $R/(x_1,\dots,x_t)$, and $\operatorname{depth}(R/(x_1,\dots,x_t)) = 0$. In particular, $\mathfrak{m} \in \operatorname{Ass}(R/(x_1,\dots,x_t))$, so there exists some $r \notin (x_1,\dots,x_t)$ such that $\mathfrak{m}r \subseteq (x_1,\dots,x_t)$. The map

$$(R/(x_1,\dots,x_t))^{\beta_n} \longrightarrow (R/(x_1,\dots,x_t))^{\beta_{n-1}}$$

is given by multiplication by a matrix whose entries are all in $\mathfrak{m}$, so is nonzero, meaning $\operatorname{Tor}_n^R(R/(x_1,\dots,x_t),M) \neq 0$.

So we have shown the theorem holds in two situations: when $\operatorname{depth}(R) = 0$ and when $\operatorname{depth}(M) = 0$. So now we assume that both $t = \operatorname{depth}(R) > 0$ and $n = \operatorname{pdim}(M) > 0$, and assume we have shown the theorem holds when $\operatorname{depth}(R) \leqslant t-1$ and $\operatorname{pdim}(M) \leqslant n-1$.

By Prime Avoidance, Lemma 3.29 from Commutative Algebra I, we can find $x \in \mathfrak{m}$ that avoids both the associated primes of M and R, so x is both regular on M and on R. By Lemma 2.14,

$$\operatorname{pdim}_{R/(x)}(M/(x)) = \operatorname{pdim}_R(M).$$

Now notice that we picked x to be regular on M, so that $\operatorname{depth}(M/xM) = \operatorname{depth}(M) - 1$. Similarly, x is regular on R, so $\operatorname{depth}(R/(x)) = \operatorname{depth}(R) - 1$. Using the assumption that the formula holds for $M/(x)$ over $R/(x)$, we conclude that

$$\begin{aligned} &\operatorname{depth}(M/(x)M) + \operatorname{pdim}_{R/(x)}(M/(x)M) = \operatorname{depth}(R/(x)) \\ \iff\ &\operatorname{depth}(M) - 1 + \operatorname{pdim}_R(M) = \operatorname{depth}(R) - 1 \\ \iff\ &\operatorname{depth}(M) + \operatorname{pdim}_R(M) = \operatorname{depth}(R). \quad \square \end{aligned}$$

This formula is very useful. For example, when doing explicit computations, it is often easier to compute a minimal free resolution for M than to compute its depth. If we happen to know $\operatorname{depth}(R)$, one can deduce $\operatorname{depth}(M)$ by computing $\operatorname{pdim}_R(M)$.

Remark 2.16. One of the consequences of Theorem 2.15 is that if a finitely generated R-module M has finite projective dimension, then

$$\operatorname{pdim}(M) \leqslant \operatorname{depth}(R) \leqslant \dim R.$$

Remark 2.17. Suppose M is a free R-module. In particular, M has finite projective dimension 0, so by the Auslander–Buchsbaum formula we have

$$\operatorname{depth}(M) = \operatorname{depth}(R).$$

Remark 2.18. Another consequence of the Auslander–Buchsbaum formula is that over a noetherian local ring of depth zero, every finitely generated module is either free or has infinite projective dimension.

Remark 2.19. Let $(R, \mathfrak{m})$ be a noetherian local ring and let $x \in \mathfrak{m}$ be a regular element. The short exact sequence

$$0 \longrightarrow R \xrightarrow{x} R \longrightarrow R/(x) \longrightarrow 0$$

is an augmented minimal free resolution for $R/(x)$, so we must have $\operatorname{pdim}(R/(x)) = 1$. By the Auslander–Buchsbaum formula,

$$\operatorname{depth}(R/(x)) = \operatorname{depth}(R) - 1.$$

Of course this also follows immediately from the fact that all maximal regular sequences have the same length, as any maximal regular sequence of length n on R that contains x gives a maximal regular sequence of length $n-1$ on $R/(x)$, and lifting any maximal regular sequence of length $n-1$ on $R/(x)$ back to R will give us a regular sequence on R of length n.

There is also a graded version of the Auslander–Buchsbaum formula.

Definition 2.20. Let $(R, \mathfrak{m})$ be a graded ring. For any finitely generated graded R-module M, we set $\operatorname{depth}(M) = \operatorname{depth}_{\mathfrak{m}}(M)$, and call it simply the **depth** of M.

Remark 2.21. By Construction 2.6, we can always pick a maximal regular sequence on M that consists entirely of homogeneous elements. But all maximal regular sequences have the same length, by Theorem 2.8, so we can also describe $\operatorname{depth}(M)$ as the maximal length of a homogeneous regular sequence on M.

Remark 2.22. Let $(R, \mathfrak{m})$ be a graded ring, as in Setting 1, and let M be a nonzero finitely generated graded R-module. First, we claim that $\mathfrak{m}$ must be in the support of M. Indeed, by Theorem 6.10 from Commutative Algebra I, $\operatorname{Supp}(M)$ is the set of primes containing $\operatorname{ann}(M)$, and one can show that the annihilator of M must be a homogeneous ideal, and thus contained in $\mathfrak{m}$.

Now consider a minimal graded resolution (F, ∂) for M over R. Localizing at $\mathfrak{m}$ gives us a free resolution for $M_{\mathfrak{m}}$, which remains minimal as the image of $\partial_{\mathfrak{m}}$ is contained in $\mathfrak{m}_{\mathfrak{m}} F_{\mathfrak{m}}$. This shows that

$$\operatorname{pdim}_{R_{\mathfrak{m}}}(M_{\mathfrak{m}}) = \operatorname{pdim}_R(M).$$

The Ext modules $\operatorname{Ext}^i_R(R/\mathfrak{m}, M)$ inherit a graded structure, and thus

$$\operatorname{Ext}^i_R(R/\mathfrak{m}, M) = 0 \iff \left(\operatorname{Ext}^i_R(R/\mathfrak{m}, M)\right)_{\mathfrak{m}} = 0.$$

By Theorem 6.35 from Homological Algebra, these Ext modules localize well, and

$$\left(\operatorname{Ext}^i_R(R/\mathfrak{m}, M)\right)_{\mathfrak{m}} \cong \operatorname{Ext}^i_{R_{\mathfrak{m}}}(R_{\mathfrak{m}}/\mathfrak{m}_{\mathfrak{m}}, M_{\mathfrak{m}}).$$

Therefore,

$$\operatorname{Ext}^i_R(R/\mathfrak{m}, M) = 0 \iff \operatorname{Ext}^i_{R_{\mathfrak{m}}}(R_{\mathfrak{m}}/\mathfrak{m}_{\mathfrak{m}}, M_{\mathfrak{m}}) = 0.$$

By Theorem 2.10, the vanishing of these Ext modules measures the depth of M over R and of $M_{\mathfrak{m}}$ over $R_{\mathfrak{m}}$. We conclude that $\operatorname{depth}(M)$ over R and $\operatorname{depth}(M_{\mathfrak{m}})$ over $R_{\mathfrak{m}}$ agree.

We can now deduce graded versions of the local results in this chapter. For example, the Auslander–Buchsbaum also holds in the graded setting:

$$\operatorname{depth}(M) + \operatorname{pdim}_R(M) = \operatorname{depth}(R).$$

Moreover, $\operatorname{depth}(R) \leqslant \operatorname{height}(\mathfrak{m}) \leqslant \dim(R)$.

We can also compare the depth of any module with its dimension. To prove that, we will need the following:

Theorem 2.23 (Ischebeck's Theorem)**.** *Let $(R, \mathfrak{m})$ be a noetherian local ring and M and N finitely generated R-modules. Then $\operatorname{Ext}^i_R(M, N) = 0$ for all $i < \operatorname{depth}(N) - \dim(M)$.*

Proof. We will give a proof by induction on $\dim(M)$, combined with the following reduction: we claim that we can reduce the problem to all modules of the form R/P with P prime.

Let M be any finitely generated R-module and assume we have already proved the statement for all modules of the form R/P with P a prime and $\dim(R/P) \leqslant \dim(M)$. Fix a prime filtration of M, meaning we get an ascending chain of submodules

$$0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_n = M$$

such that $M_i/M_{i-1} \cong R/P_i$ for some primes P_i. Such a prime filtration exists by Theorem 6.33 from Commutative Algebra I. Notice also that by the construction in Theorem 6.33 from Commutative Algebra I, all the P_i contain $\operatorname{ann}(M)$, so $\dim(R/P_i) \leqslant \dim(M)$.

Our assumption is that $\operatorname{Ext}^j_R(R/P_i, N) = 0$ for all $j < \operatorname{depth}(N) - \dim(R/P_i)$ for each P_i. But $\dim(R/P_i) \leqslant \dim(M)$, so in particular for all i we know that

$$\operatorname{Ext}^j_R(M_i/M_{i-1}, N) = 0 \quad \text{for } j < \operatorname{depth}(N) - \dim(M).$$

In particular, since $R/P_1 \cong M_1/M_0 \cong M_1$, we know that

$$\operatorname{Ext}^j_R(M_1, N) = 0 \quad \text{for } j < \operatorname{depth}(N) - \dim(M).$$

Now for each $i \geqslant 1$, break the filtration into short exact sequences

$$0 \longrightarrow M_{i-1} \longrightarrow M_i \longrightarrow R/P_i \longrightarrow 0.$$

Look at the long exact sequence we get when we apply $\mathrm{Hom}_R(-, N)$:

$$\cdots \to \mathrm{Ext}_R^{j-1}(R/P_i, N) \to \mathrm{Ext}_R^j(M_i, N) \to \mathrm{Ext}_R^j(M_{i-1}, N) \to \mathrm{Ext}_R^j(R/P_i, N) \to \cdots.$$

If $j = 0 < \mathrm{depth}(N) - \dim(M)$, then $\mathrm{Ext}_R^0(R/P_i, N) = 0$ and the long exact sequence gives us

$$0 \longrightarrow \mathrm{Ext}_R^0(M_i, N) \longrightarrow \mathrm{Ext}_R^0(M_{i-1}, N) \longrightarrow 0,$$

and thus an isomorphism $\mathrm{Ext}_R^j(M_i, N) \cong \mathrm{Ext}_R^j(M_{i-1}, N)$. For $1 \leqslant j < \mathrm{depth}(N) - \dim(M)$, we know that $\mathrm{Ext}_R^j(R/P_i, N) = 0 = \mathrm{Ext}_R^{j-1}(R/P_i, N)$, so we also get an isomorphism

$$\mathrm{Ext}_R^j(M_i, N) \cong \mathrm{Ext}_R^j(M_{i-1}, N).$$

But $\mathrm{Ext}_R^j(M_1, N) = 0$, so inductively we conclude that $\mathrm{Ext}_R^j(M_i, N) = 0$ for all i. In particular, since $M = M_n$, we conclude that

$$\mathrm{Ext}_R^j(M_i, N) = 0 \text{ for all } j < \mathrm{depth}(N) - \dim(M).$$

Now that we have this reduction at our disposal, we proceed to prove the statement for a general M by induction. When $\dim(M) = 0$, the reduction argument above tells us we need only to consider the case when $M = R/P$ and $\dim(R/P) = 0$. In that case, we must have $P = \mathfrak{m}$, so by Theorem 2.10 we have $\mathrm{Ext}_R^i(R/\mathfrak{m}, N) = 0$ for all $i < \mathrm{depth}(N) = \mathrm{depth}(N) - \dim(R/P)$.

Suppose that for some $d \geqslant 1$, we have proved the statement holds for all R-modules of dimension $d-1$, and let $\dim(M) = d$. Again, we need only to prove the statement holds for $M = R/P$. Since $\dim(R/P) > 0$, then $P \neq \mathfrak{m}$, so we can pick $x \in \mathfrak{m}$ with $x \notin P$. Since the image of x in the domain R/P is nonzero, we know by Exercise 12 that

$$\dim(R/(P + (x)) = \dim(R/P) - 1.$$

By induction hypothesis,[2] we have $\mathrm{Ext}_R^i(R/(P + (x)), N) = 0$ for all

$$i < \mathrm{depth}(N) - \dim(R/P + (x)) = \mathrm{depth}(N) - \dim(R/P) + 1.$$

Moreover, the short exact sequence

$$0 \longrightarrow R/P \xrightarrow{x} R/P \longrightarrow R/(P + (x)) \longrightarrow 0$$

gives rise to the long exact sequence

$$\cdots \longrightarrow \mathrm{Ext}_R^i\left(\tfrac{R}{P+(x)}, N\right) \longrightarrow \mathrm{Ext}_R^i(R/P, N) \xrightarrow{x} \mathrm{Ext}_R^i(R/P, N) \longrightarrow \mathrm{Ext}_R^{i+1}\left(\tfrac{R}{P+(x)}, N\right) \longrightarrow \cdots.$$

[2]Note that this is the step that required us to set up our induction so carefully, as $P + (x)$ is not necessarily a prime ideal anymore.

Therefore, for all $i < \operatorname{depth}(N) - \dim(R/P)$ we have both

$$\operatorname{Ext}^i_R(R/(P+(x)), N) = 0 \operatorname{Ext}^{i+1}_R(R/P+(x), N).$$

Thus the exact sequence above gives us an isomorphism

$$\operatorname{Ext}^i_R(R/P, N) \xrightarrow{x} \operatorname{Ext}^i_R(R/P, N).$$

In particular, $\operatorname{Ext}^i_R(R/P, N) = x \cdot \operatorname{Ext}^i_R(R/P, N)$. Moreover, by Theorem 6.36 from Homological Algebra, the module $\operatorname{Ext}^i_R(R/P, N)$ is finitely generated, so $\operatorname{Ext}^i_R(R/P, N) = 0$ by NAK (see Theorem 5.32 from Commutative Algebra I). This completes the proof of the case $M = R/P$ for a prime ideal P. □

We already know that $\operatorname{depth}(R) \leqslant \dim(R)$. We can now obtain a generalization of this inequality for all R-modules.

Corollary 2.24. *Let $(R, \mathfrak{m})$ be a noetherian local ring and M a finitely generated R-module. For every associated prime P of M, $\operatorname{depth}(M) \leqslant \dim(R/P)$. In particular,*

$$\operatorname{depth}(M) \leqslant \dim(M).$$

Proof. Let P be an associated prime of M. By Theorem 2.23, $\operatorname{Ext}^i_R(R/P, M) = 0$ for all $i < \operatorname{depth}(M) - \dim(R/P)$. But every element in P is a zerodivisor on M, so $\operatorname{depth}_P(M) = 0$. Moreover, by Theorem 2.10, $\operatorname{Ext}^0_R(R/P, M) \neq 0$. We conclude that $\operatorname{depth}(M) \leqslant \dim(R/P)$. Finally, note that

$$\begin{aligned}\dim(M) &= \max\{\dim(R/P) \mid P \in \operatorname{Min}(M)\} \\ &= \max\{\dim(R/P) \mid P \in \operatorname{Ass}(M)\}\end{aligned}$$

so $\operatorname{depth}(M) \leqslant \dim(M)$. □

Next, we prove a useful lemma we will use later:

Lemma 2.25. *Let $(R, \mathfrak{m})$ be a noetherian local ring and M a finitely generated R-module. Given any ideal I in R, $\operatorname{depth}_I(M) = \operatorname{depth}_{\sqrt{I}}(M)$.*

Proof. On the one hand, $I \subseteq \sqrt{I}$, so

$$\operatorname{depth}_I(M) \leqslant \operatorname{depth}_{\sqrt{I}}(M).$$

On the other hand, if $x_1, \dots, x_n \in \sqrt{I}$ is a maximal regular sequence on M inside, then there exists $a_1, \dots, a_n > 0$ such that $x_1^{a_1}, \dots, x_n^{a_n} \in I$, but by Lemma 1.52 $x_1^{a_1}, \dots, x_n^{a_n}$ is a regular sequence on M. □

Exercise 28. Let R be a noetherian ring and W a multiplicatively closed subset. Show that if $x_1, \dots, x_n \notin W$ form a regular sequence in R, then $\frac{x_1}{1}, \dots, \frac{x_n}{1}$ is a regular sequence on $W^{-1}R$.

Exercise 29. Let R be a noetherian ring, and let P be a prime ideal containing an ideal I.

1) Show that $\operatorname{grade}(I_P) \geqslant \operatorname{grade}(I)$.

2) Give an example where $\operatorname{grade}(I_P) > \operatorname{grade}(I)$.

2.2 Cohen-Macaulay rings

> Life is really worth living in a noetherian ring R when all the local rings have the property that every system of parameters is an R-sequence. Such a ring is called Cohen-Macaulay (C-M for short).
>
> Mel Hochster, page 887 of [Hoc78]

Irvin Cohen and Francis Macaulay were two big influences in the early days of commutative algebra. Cohen-Macaulay rings, named after the two of them, are by some measure the largest class of nice rings commutative algebraists study. They are on the border of being just nice enough to make life is easier, and just broad enough to contain many interesting examples. One of the main reference books in any commutative algebraist's shelf is dedicated to Cohen-Macaulay rings specifically [BH93]. In this section, we will see some of the reasons why life really is worth living in a Cohen-Macaulay ring.

Given a noetherian local ring R,

$$\operatorname{depth}(R) \leqslant \dim(R) \leqslant \operatorname{embdim}(R).$$

When the second inequality is an equality, we have a regular ring. When the first inequality is an equality, our ring is Cohen-Macaulay.

More generally, for an R-module M, we proved in Corollary 2.24 that

$$\operatorname{depth}(M) \leqslant \dim(M).$$

When equality happens, we say M is a Cohen-Macaulay module.

Definition 2.26. A noetherian local ring R is **Cohen-Macaulay** if $\operatorname{depth}(R) = \dim(R)$. More generally, an R-module M is **Cohen-Macaulay** if $\operatorname{depth}(M) = \dim(M)$. We say that M is a **maximal Cohen-Macaulay module**, or **MCM module**, if $\operatorname{depth}(M) = \dim(R)$.

Remark 2.27. A local ring is Cohen-Macaulay if it is a Cohen-Macaulay module over itself.

Example 2.28. All artinian rings are Cohen-Macaulay, as the inequalities

$$0 = \dim(R) \geqslant \operatorname{depth}(R)$$

automatically imply $\operatorname{depth}(R) = 0$.

Theorem 2.29. *Regular local rings are Cohen-Macaulay.*

Proof. Let $(R, \mathfrak{m})$ be a regular local ring. By Auslander–Buchsbaum–Serre, Theorem 1.71, $\mathfrak{m}$ is generated by a regular sequence of length $\dim(R)$, so $\operatorname{depth}(R) = \dim(R)$. □

But not all Cohen-Macaulay rings are regular.

Example 2.30. The artinian local ring $k[x]/(x^2)$ is Cohen-Macaulay but not regular.

Example 2.31. Every 1-dimensional domain is Cohen-Macaulay, since any nonzero nonunit is a regular element.

Exercise 30. Show that $R = k[\![x, y, z]\!]/(xy, xz)$ is not Cohen-Macaulay.

Exercise 31. Let $R = k[\![x, y]\!]/(x^2, xy)$ and $M = R/(x)$. Show M is an MCM module but R is not Cohen-Macaulay.

Remark 2.32. If M is a finitely generated R-module of finite projective dimension, then by the Auslander–Buchsbaum formula, Theorem 2.15, we have $\operatorname{depth}(M) \leqslant \operatorname{depth}(R)$. If M is a finitely generated MCM module of finite projective dimension, then the inequalities

$$\operatorname{depth}(M) \leqslant \operatorname{depth}(R) \leqslant \dim(R)$$

must be equalities throughout, and thus R itself must be Cohen-Macaulay. Moreover, by the Auslander–Buchsbaum formula, Theorem 2.15,

$$\operatorname{pdim}(M) = \operatorname{depth}(R) - \operatorname{depth}(M) = 0.$$

Thus M must be free.

Theorem 2.33. *Let $(R, \mathfrak{m}, k)$ be a noetherian local ring and let $\underline{x} = x_1, \dots, x_n$ be a regular sequence. The ring R is Cohen-Macaulay if and only if $R/(\underline{x})$ is Cohen-Macaulay.*

Proof. It is sufficient to prove the case when $n = 1$, since we can then apply the statement repeatedly. Let x be a regular element on R. By Exercise 12, $\dim(R/(x)) = \dim(R) - 1$. By Remark 2.19, $\operatorname{depth}(R/(x)) = \operatorname{depth}(R) - 1$. Thus R is Cohen-Macaulay if and only if $R/(x)$ is Cohen-Macaulay. □

Many rings with *nice* singularities are Cohen-Macaulay. For example, Hochster and Roberts famously showed [HR74] that rings of invariants of any finite group G over a field k of characteristic not dividing $|G|$ are Cohen-Macaulay. Their proof used prime characteristic techniques, introducing what is now a very important class of characteristic p singularities, and which are essentially homological in nature.

Hochster's quote above pointed us to another characterization of Cohen-Macaulayness, for which we will need to recall the notion of a system of parameters.

Definition 2.34. Let $(R, \mathfrak{m})$ be a noetherian local ring of dimension d. A sequence of d elements $x_1, \dots, x_d$ is a **system of parameters** or **SOP** if $\sqrt{(x_1, \dots, x_d)} = \mathfrak{m}$. We say that elements $x_1, \dots, x_t$ are **parameters** if they are part of a system of parameters.

In the graded case, a **homogeneous system of parameters** is a system of parameters consisting only of homogeneous elements.

By Theorem 8.13 from Commutative Algebra I, every local (or graded) ring admits a system of parameters, and these can be useful in characterizing the dimension of a noetherian local ring, or the height of a prime in a noetherian ring. Most importantly to our current endeavours, we can characterize Cohen-Macaulayness in terms of SOPs.

Theorem 2.35. *Let $(R, \mathfrak{m})$ be a noetherian local ring. The following are equivalent:*

1) R is Cohen-Macaulay.

2) Some system of parameters in R is a regular sequence on R.

3) Every system of parameters in R is a regular sequence on R.

Proof. The implications 3 $\Rightarrow$ 2 $\Rightarrow$ 1 are obvious.

To show 1 $\Rightarrow$ 3, suppose R is Cohen-Macaulay and let $\underline{x} = x_1, \dots, x_d \in R$ be a system of parameters, meaning $\sqrt{(\underline{x})} = \mathfrak{m}$. By Lemma 2.25 and the fact that R is Cohen-Macaulay,

$$\operatorname{depth}_{(\underline{x})}(R) = \operatorname{depth}_{\mathfrak{m}}(R) = d.$$

By the depth sensitivity of the Koszul complex, Theorem 2.11, $\mathrm{H}_i(\underline{x}) = 0$ for all $i \geqslant 1$. By Theorem 1.49, this implies that $\underline{x}$ is a regular sequence. □

Theorem 2.36. *Let $(R, \mathfrak{m})$ be a noetherian local ring. For all Cohen-Macaulay R-modules M and all $P \in \operatorname{Ass}(M)$,*

$$\operatorname{depth}(M) = \dim(R/P).$$

In particular, if R is a Cohen-Macaulay local ring, then R has no embedded primes, and all maximal chains of primes in R have the same length: for all $P \in \operatorname{Min}(R) = \operatorname{Ass}(R)$,

$$\dim(R/P) = \dim(R).$$

Proof. By the inequality on depth and dimension of associated primes from Corollary 2.24,

$$\dim(M) = \max\{\dim(R/\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Ass}(M)\} \geqslant \min\{\dim(R/\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Ass}(M)\} \geqslant \operatorname{depth}(M).$$

Since M is Cohen-Macaulay, equality holds throughout, so $\dim(M) = \dim(R/P)$ for all $P \in \operatorname{Ass}(M)$. □

Theorem 2.36 implies that Cohen-Macaulay rings are equidimensional.

We saw in Remark 2.3 that we always have $\operatorname{grade}(I) \leqslant \operatorname{height}(I)$. We will be proving that equality holds for any ideal in a Cohen-Macaulay ring. We start by proving this for prime ideals, which will allow us to show that the Cohen-Macaulay property localizes.

Theorem 2.37. *Every prime ideal P a Cohen-Macaulay local ring has* $\operatorname{grade}(P) = \operatorname{height}(P)$.

Proof. By Remark 2.3, we know that $\operatorname{grade}(P) \leqslant \operatorname{height}(P)$. We will show by induction on the height of P that there is a regular sequence contained in P of length equal to the height of P. If P is a minimal prime, there is nothing to show.

Suppose that for some $h \geqslant 1$, the statement holds for all primes P of height $h-1$ in any Cohen-Macaulay local ring. Let P be a prime of height h in a Cohen-Macaulay local ring R. Since P is not minimal, it is not contained in the union of the minimal primes, hence not in the union of the associated primes by Theorem 2.36. Thus, there is a regular element $x \in P$.

By Theorem 2.33, $R/(x)$ is a Cohen-Macaulay ring, and $\operatorname{height}(P/(x)) = \operatorname{height}(P) - 1 = h - 1$. By induction hypothesis, $P/(x)$ contains a regular sequence $\underline{y}$ of length $h-1$, and thus P contains the regular sequence $x, \underline{y}$ of length h. □

We are now ready to show that the Cohen-Macaulay property localizes.

Theorem 2.38. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring and P be a prime ideal in R. Then R_P is a Cohen-Macaulay local ring.*

Proof. By Theorem 2.37, $\operatorname{grade}(P) = \operatorname{height}(P)$. By Exercise 29, $\operatorname{grade}(P_P) \geqslant \operatorname{grade}(P)$. On the other hand, grade is always bounded above by height, by Remark 2.3, so

$$\operatorname{height}(P) = \operatorname{height}(P_P) \geqslant \operatorname{grade}(P_P) \geqslant \operatorname{grade}(P) = \operatorname{height}(P).$$

We conclude that

$$\operatorname{depth}(R_P) = \operatorname{grade}(P_P) = \operatorname{height}(P) = \dim(R_P)$$

and thus R_P is Cohen-Macaulay. □

We can now give a global definition of Cohen-Macaulayness.

Definition 2.39. A noetherian ring R is Cohen-Macaulay if R_P is Cohen-Macaulay for all primes P. An R-module M is Cohen-Macaulay if M_P is Cohen-Macaulay for all primes P.

Remark 2.40. Since localizations of local Cohen-Macaulay rings are Cohen-Macaulay by Theorem 2.38, to check that a ring R is Cohen-Macaulay it suffices to check that $R_{\mathfrak{m}}$ is Cohen-Macaulay for all maximal ideals $\mathfrak{m}$.

Corollary 2.41. *All regular rings are Cohen-Macaulay. In particular, any polynomial ring over a field is Cohen-Macaulay.*

Theorem 2.42 (Dimension formula)**.** *Let R be a Cohen-Macaulay ring, and let $P \subseteq Q$ be primes. Then*

$$\operatorname{height}(Q) - \operatorname{height}(P) = \dim(R_Q/PR_Q).$$

Proof. The localization R_Q is a Cohen-Macaulay local ring of dimension $\operatorname{height}(Q)$. Set

$$h = \operatorname{height}(P) = \operatorname{height}(PR_Q).$$

By Theorem 2.37, we can pick h many elements $x_1, \dots, x_h \in P$ such that their image in R_Q is a regular sequence inside PR_Q.

Therefore, $R_Q/(x_1, \dots, x_h)R_Q$ is Cohen-Macaulay by Theorem 2.33, and by Exercise 12

$$\dim(R_Q/(x_1, \dots, x_h)R_Q) = \dim(R_Q) - h = \operatorname{height}(Q) - \operatorname{height}(P).$$

By Theorem 1.55, $\operatorname{height}(x_1, \dots, x_h) = h$, and since P has height h and contains $(x_1, \dots, x_h)$, P must be a minimal prime over $(x_1, \dots, x_h)$. Since $R_Q/(x_1, \dots, x_h)R_Q$ is Cohen-Macaulay, we know by Theorem 2.36 that

$$\dim(R_Q/(x_1, \dots, x_h)R_Q) = \dim(R_Q/PR_Q).$$

We conclude that

$$\operatorname{height}(Q) - \operatorname{height}(P) = \dim(R_Q/PR_Q) = \dim(R_Q/(x_1, \dots, x_h)R_Q).$$ □

Theorem 2.43. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring, and let I be any proper ideal in R. Then*

$$\dim(R/I) = \dim(R) - \text{height}(I).$$

Proof. Applying Theorem 2.42 to $Q = \mathfrak{m}$ and any prime ideal P, and noticing that $R_{\mathfrak{m}} = R$, gives us

$$\dim(R) - \text{height}(P) = \text{height}(\mathfrak{m}) - \text{height}(P) = \dim(R/P).$$

Hence, the formula holds when $I = P$ is prime.

For an arbitrary ideal I, fix a minimal prime P of I such that $\text{height}(P) = \text{height}(I)$. Note that P/I is a minimal prime of R/I, so

$$\dim(R/I) \geqslant \dim(R/P).$$

Then

$$\dim(R/I) + \text{height}(I) \geqslant \dim(R/P) + \text{height}(I) = \dim(R/P) + \text{height}(P) = \dim(R).$$

On the other hand, if Q is a minimal prime of I with $\dim(R/I) = \dim(R/Q)$, then $\text{height}(I) \leqslant \text{height}(Q)$, and

$$\dim(R/I) + \text{height}(I) \leqslant \dim(R/I) + \text{height}(Q) = \dim(R/Q) + \text{height}(Q) = \dim(R). \quad \square$$

Definition 2.44. An ideal I is **unmixed** if it has no embedded primes, that is, if all its associated primes are minimal. An ideal I is **height unmixed** if every associated prime of I has the same fixed height.

Note that height unmixed implies unmixed.

Lemma 2.45. *In a Cohen-Macaulay local ring $(R, \mathfrak{m})$, all ideals generated by a regular sequence are height unmixed: given a regular sequence $x_1, \ldots, x_n$, every associated prime of $(x_1, \ldots, x_n)$ must have height n.*

Proof. Let P be an associated prime of $(x_1, \ldots, x_n)$. By Theorem 6.38 from Commutative Algebra I, P_P is an associated prime of $(x_1, \ldots, x_n)R_P$.

By Theorem 2.38, R_P is Cohen-Macaulay. Moreover, the images of $x_1, \ldots, x_n$ in R_P also form a regular sequence in R_P, by Exercise 29, so by Theorem 2.33, $R_P/(\underline{x})R_P$ is also a Cohen-Macaulay ring. By Theorem 2.36, Cohen-Macaulay rings have no embedded primes, so P_P is a minimal prime of $R_P/(\underline{x})_P$, and thus P is a minimal prime of $(\underline{x})$.

By Krull's Height Theorem, Theorem 8.5 from Commutative Algebra I, $\text{height}(P) \leqslant n$. By Remark 2.3, $\text{height}(P) \geqslant \text{grade}(P) \geqslant n$, so $\text{height}(P) = n$. $\square$

Theorem 2.46. *Let R be a noetherian ring. The following are equivalent:*

1) *R is Cohen-Macaulay.*

2) *For all ideals I, $\operatorname{grade}(I) \geqslant \operatorname{height}(I)$.*

3) *For all ideals I, $\operatorname{grade}(I) = \operatorname{height}(I)$.*

Proof. We proved in Remark 2.3 that $\operatorname{grade}(I) \leqslant \operatorname{height}(I)$ for all ideals I, so 2 $\Rightarrow$ 3.

To show 3 $\Rightarrow$ 1, consider a maximal ideal $\mathfrak{m}$ in R. By assumption, $\operatorname{grade}(\mathfrak{m}) = \operatorname{height}(\mathfrak{m})$, and by Exercise 29 $\operatorname{grade}(\mathfrak{m}_{\mathfrak{m}}) \geqslant \operatorname{grade}(\mathfrak{m})$. Therefore,

$$\operatorname{depth}(R_{\mathfrak{m}}) = \operatorname{grade}(\mathfrak{m}_{\mathfrak{m}}) \geqslant \operatorname{grade}(\mathfrak{m}) = \operatorname{height}(\mathfrak{m}) = \dim(R_{\mathfrak{m}}).$$

Thus $R_{\mathfrak{m}}$ is Cohen-Macaulay for all maximal ideals $\mathfrak{m}$, and so R must be Cohen-Macaulay.

To show 1 $\Rightarrow$ 3, let $n = \operatorname{grade}(I)$ and let $x_1, \dots, x_n$ be a maximal regular sequence inside I. The elements of I are all be zerodivisors on $R/(x_1, \dots, x_n)$, by maximality, so by Lemma 2.5 the ideal I must be contained in some associated prime P of $R/(x_1, \dots, x_n)$. By Lemma 2.45, P has height n, so

$$n = \operatorname{grade}(I) \leqslant \operatorname{height}(I) \leqslant \operatorname{height}(P) = n. \qquad \square$$

Corollary 2.47. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring, and let I be a proper ideal in R. The following are equivalent:*

1) *The ideal I is generated by a regular sequence.*

2) *Every minimal generating set for I is a regular sequence.*

3) *$\mu(I) = \operatorname{height}(I)$.*

Proof. By Theorem 2.46, $\operatorname{grade}(I) = \operatorname{height}(I)$. The implication 2 $\Rightarrow$ 1 is obvious.

If I is generated by a regular sequence, then that regular sequence must have length $\operatorname{height}(I)$, so I must be generated by exactly $\operatorname{height}(I)$ elements, proving $\mu(I) \leqslant \operatorname{height}(I)$. On the other hand, the inequality $\mu(I) \geqslant \operatorname{height}(I)$ always holds by Krull's Height Theorem, Theorem 8.5 from Commutative Algebra I, so $\mu(I) = \operatorname{height}(I)$. This proves 1 $\Rightarrow$ 3.

To show 3 $\Rightarrow$ 1, suppose that $\mu(I) = \operatorname{height}(I) = h$, and consider any set of minimal generators $x_1, \dots, x_h$ for I. By Theorem 2.11, since $\operatorname{grade}(I) = \operatorname{height}(I)$ we must have $\mathrm{H}_i(\underline{x}) = 0$ for all $i \geqslant 1$. By Theorem 1.49, the vanishing of koszul homology tells us that $\underline{x}$ itself must be a regular sequence. $\square$

By Lemma 2.45 and Corollary 2.47, in a Cohen-Macaulay local ring every ideal I generated by $\operatorname{height}(I)$ many elements is unmixed. In fact, this characterizes Cohen-Macaulay rings even in the nonlocal case.

Theorem 2.48 (Unmixedness Theorem)**.** *A noetherian ring R is Cohen-Macaulay if and only if every ideal $I = (a_1, \dots, a_n)$ with* $\operatorname{height}(I) = n$ *is unmixed.*

Proof. ($\Rightarrow$) Let R be Cohen-Macaulay and $I = (x_1, \dots, x_n)$ of $\operatorname{height}(I) = n$. Fix $P \in \operatorname{Ass}(I)$. By Theorem 2.38, R_P is Cohen-Macaulay. Therefore, $\operatorname{grade}(I_P) = \operatorname{height}(I_P) = n$. Since I_P is generated by the n elements $\frac{a_1}{1}, \dots, \frac{a_n}{1}$, by Corollary 2.47 we conclude that $\frac{a_1}{1}, \dots, \frac{a_n}{1}$ is a regular sequence on R_P. Therefore, IR_P is generated by a regular sequence, and thus by Lemma 2.45 it has no embedded primes. By Theorem 6.38 from Commutative Algebra I, P_P is an associated prime of $(x_1, \dots, x_n)R_P$, and thus P_P must be a minimal prime of $(x_1, \dots, x_n)R_P$. We conclude that P itself is a minimal prime of I.

($\Leftarrow$) Let P be a prime ideal of height h. By Theorem 8.20 from Commutative Algebra I, we can find $x_1, \dots, x_h \in P$ such that P a minimal prime over $(x_1, \dots, x_h)$. In fact, we can choose $x_1, \dots, x_h$ such that $\operatorname{height}(x_1, \dots, x_n) = n$ for all $n \leqslant h$. By assumption, $(x_1, \dots, x_n)$ has no embedded primes for each n. Since x_{n+1} is not in any minimal prime of $R/(x_1, \dots, x_n)$ for each n, it must in fact be regular on $R/(x_1, \dots, x_n)$. We conclude that $x_1, \dots, x_h$ is a regular sequence.

By Exercise 29, $\frac{x_1}{1}, \dots, \frac{x_n}{1} \in PR_P$ is also a regular sequence. Since $\dim(R_P) = \operatorname{height}(P) = h$, we conclude that R_P is Cohen-Macaulay. □

Theorem 2.49. *Let $A \subseteq R$ be a module-finite extension of noetherian local rings. If A is a regular local ring, then R is Cohen-Macaulay if and only if R is free as an A-module.*

Proof. Module-finite extensions are integral, and thus $d = \dim(R) = \dim(A) = \operatorname{depth}(A)$. Moreover, if $x_1, \dots, x_d \in A$ is a system of parameters for A, then we claim $x_1, \dots, x_d$ is also a system of parameters for R. To see that, set $\mathfrak{m}_R$ and $\mathfrak{m}_A$ to be the maximal ideals pf A and R, respectively, and note that $\mathfrak{m}_R \cap A = \mathfrak{m}_A$. By Incomparability, Theorem 7.32 from Commutative Algebra I, $\mathfrak{m}_R$ must be the unique prime in R containing $x_1, \dots, x_d)R$, so $\sqrt{(x_1, \dots, x_d)R} = \mathfrak{m}_R$. Since $\dim(R) = d$, we conclude that $x_1, \dots, x_d$ is a system of parameters for R.

Suppose that as an A-module, R has depth d. Any sequence of elements in A that are regular on R is also a sequence of elements in R that are regular on R, so the depth of R as an R-module is also $d = \dim(R)$, and thus R is a Cohen-Macaulay ring. Conversely, if R is a Cohen-Macaulay ring, then any system of parameters for A is also a system of parameters for R, and thus a regular sequence on R. In particular, the depth of R as an A-module is d. We have thus shown that the depth of R as an A-module satisfies the following:

$$R \text{ is a Cohen-Macaulay ring} \iff \operatorname{depth}(R) = d.$$

Since A is regular, by Auslander–Buchsbaum–Serre, Theorem 1.71, R has finite projective dimension as an A-module, so the Auslander–Buchsbaum formula, Theorem 2.15, applies to R as an A-module. Thus

$$R \text{ is Cohen-Macaulay} \iff \operatorname{depth}(R) = d \iff \operatorname{pdim}_A(R) = 0 \iff R \text{ if free over } A. \quad \square$$

Next, we consider Cohen-Macaulayness in the graded setting.

Lemma 2.50. *Let $(R, \mathfrak{m})$ be a graded finitely generated k-algebra. If $R_{\mathfrak{m}}$ is Cohen-Macaulay, then every homogeneous system of parameters in R is a regular sequence.*

Proof. Let $\underline{x} = x_1, \dots, x_n$ be a homogeneous system of parameters in R, so that $\sqrt{(\underline{x})} = \mathfrak{m}$. Note that its localization $\underline{y} = y_1, \dots, y_n$ with $y_i = \frac{x_i}{1}$ is a system of parameters for $R_{\mathfrak{m}}$, as

$$\sqrt{(\underline{y})} = \left(\sqrt{(\underline{x})}\right)_{\mathfrak{m}} = \mathfrak{m}_{\mathfrak{m}}.$$

Since $R_{\mathfrak{m}}$ is Cohen-Macaulay, Theorem 2.35 says that $\underline{y}$ is a regular sequence on $R_{\mathfrak{m}}$. If $\underline{x}$ is not a regular sequence on R, then for some i we would have $x_i \in P$ for some associated prime of $R/(x_1, \dots, x_{i-1})$. The associated primes of a graded module must be all homogeneous, so $P \subseteq \mathfrak{m}$. In particular, $P_{\mathfrak{m}}$ is associated to the localization

$$(R/(x_1, \dots, x_{i-1}))_{\mathfrak{m}} \cong R_{\mathfrak{m}}/(y_1, \dots, y_{i-1})$$

and $y_i \in P_{\mathfrak{m}}$. But this contradicts the fact that $\underline{y}$ is a regular sequence, so we conclude that $\underline{x}$ is a regular sequence on R. □

Definition 2.51. Let R be a standard graded finitely generated k-algebra over a field k. A **graded Noether normalization** of R is a subalgebra $A = k[x_1, \dots, x_t] \subseteq R$ with $x_1, \dots, x_t \in R$ homogeneous and algebraically independent over k and $A \subseteq R$ module-finite.

Theorem 2.52. *Let k be a field and $R = \bigoplus_{i \geqslant 0} R_i$ a graded finitely generated k-algebra with $R_0 = k$ and a graded Noether normalization S. Then R is a Cohen-Macaulay ring if and only if R is a free S-module.*

Proof. Fix algebraically independent homogeneous $x_1, \dots, x_d \in R$ with $S = k[x_1, \dots, x_d]$.

($\Rightarrow$) Suppose that R is Cohen-Macaulay with homogeneous maximal ideal $\mathfrak{m}$. Since $x_1, \dots, x_d$ are homogeneous in R, we have $(x_1, \dots, x_d)R \subseteq \mathfrak{m}$ and thus $(x_1, \dots, x_d) \subseteq \mathfrak{m} \cap R$. Since $\mathfrak{m}$ is maximal, we conclude that $\mathfrak{m} \cap R = (x_1, \dots, x_d)$. By Incomparability, Theorem 7.32 from Commutative Algebra I, $\mathfrak{m}$ must be a minimal prime over $(x_1, \dots, x_d)$ in R. Since the extension $S \subseteq R$ is integral, we know that $\dim(R) = \dim(S) = d$. Thus $x_1, \dots, x_d$ is a homogeneous system of parameters in R. Moreover, since R is Cohen-Macaulay then so is $R_{\mathfrak{m}}$, and thus $\underline{x}$ must be a regular sequence on R by Lemma 2.50. We conclude that the depth of R as a graded S-module is d. By the graded Auslander–Buchsbaum formula in Remark 2.22,

$$\operatorname{pdim}_S(R) = \operatorname{depth}(S) - \operatorname{depth}(R) = 0.$$

Therefore, R is a projective graded S-module. As we noted in Remark 1.2, bounded below graded projectives are free, and thus R must be a free S-module.

($\Leftarrow$) Suppose that R is a free S-module. Let P be any maximal ideal of R, and let $Q = P \cap S$. We need to show that R_P is Cohen-Macaulay. First, note that since $S \subseteq R$ is an integral extension, Q must also be maximal, by Lemma 7.30 from Commutative Algebra I.

Since S is regular, S_Q is Cohen-Macaulay. Since R is a free S-module, then the localization R_Q of R at Q as an S-module is a free S_Q-module. We can also view R_Q as the localization of the ring R at the multiplicatively closed subset of R given by $W = S \setminus Q$. Since $W \subseteq R \setminus P$, then R_P is a further localization of R_Q. Thus we get ring homomorphisms $S_Q \to R_Q \to R_P$ where R_Q is free as an S_Q-module, and R_P is a flat module over R_Q. We conclude that R_P is a flat S_Q-module, as

$$-\otimes_{S_Q} R_P \cong (-\otimes_{R_Q} R_P) \circ (-\otimes_{S_Q} R_Q)$$

is a composition of exact functors. This shows that R_P is a flat module over S_Q.

Fix a system of parameters $\underline{w}$ in S_Q. Since S_Q is Cohen-Macaulay, $\underline{w}$ is a regular sequence on S_Q. Thus $\operatorname{kos}(\underline{w}; S_Q)$ is exact in degrees 1 and above. By flatness,

$$\operatorname{kos}(\underline{w}; S_Q) \otimes_{S_Q} R_P \cong \operatorname{kos}(\underline{w}; R_P)$$

is also exact in degrees 1 and above, so $\underline{w}$ is a regular sequence on R_P by Theorem 1.49.

Finally, note that $\underline{w}$ has length $\dim(S_Q) = \operatorname{height}(Q)$. Since $S \subseteq R$ is integral and $Q = P \cap S$, we know that $\dim(R_P) = \operatorname{height}(P) \leqslant \operatorname{height}(Q)$, by Corollary 7.34 from Commutative Algebra I. On the other hand, since the image of $\underline{w}$ in R_P is a regular sequence on R_P of length $\operatorname{height}(Q)$, we must also have $\dim(R_P) \geqslant \operatorname{height}(Q)$. Therefore, $\underline{w}$ is a regular sequence in R_P of length $\dim(R_P)$, and R_P is a Cohen-Macaulay ring. □

Theorem 2.53. *Let k be a field and $R = \oplus_{i \geqslant 0} R_i$ be a finitely generated graded k-algebra with $R_0 = k$ and homogeneous maximal ideal $\mathfrak{m} = R_+ = \oplus_{i \geqslant 1} R_i$. If $R_{\mathfrak{m}}$ is Cohen-Macaulay, then R is Cohen-Macaulay.*

Proof. Let $\underline{x} = x_1, \ldots, x_n$ be a homogeneous system of parameters in R. By Lemma 2.50, $\underline{x}$ is a regular sequence in R. We claim that $k[\underline{x}] \subseteq R$ is an integral extension.

Since $\mathfrak{m} = \sqrt{(\underline{x})}$, there is some N such that $\mathfrak{m}^N \subseteq (\underline{x})$. In particular, the graded ring $R/(\underline{x})$ has dimension 0, as it is killed by $\mathfrak{m}^N$. Therefore, $R/(\underline{x})$ is a finite-dimensional k-vector space. Since R is a bounded below graded S-module, by graded NAK, Theorem 5.41 from Commutative Algebra I, we conclude that R is finitely generated as a module over $k[\underline{x}]$, generated by the lifts of a basis for $R/(\underline{x})R$ over k.

We claim that the homogeneous system of parameters $\underline{x}$ is algebraically independent. If not, then $k[\underline{x}] \subseteq R$ would be an integral extension with $\dim(k[\underline{x}]) < n = \operatorname{height}(\mathfrak{m}) \leqslant \dim(R)$, which is impossible. Thus $S = k[x_1, \ldots, x_n]$ is a graded Noether normalization of R, and $\dim(R) = \dim(S) = n$.

Now we claim that R is a free S-module. First, note that by construction $\underline{x} \in S$ is a regular sequence on R, so the graded depths match: $\operatorname{depth}(R) = n = \dim(S) = \operatorname{depth}(S)$. By the Auslander–Buchsbaum Formula, Theorem 2.15, which we saw in Remark 2.22 applies to this graded version of depth, we must have

$$\operatorname{pdim}_S(R) = \operatorname{depth}(S) - \operatorname{depth}(R) = 0.$$

Therefore, R is a projective S-module. As noted in Remark 1.2, bounded below graded projectives are free, so R is a free S-module. By Theorem 2.52, R is Cohen-Macaulay. □

Remark 2.54. Let k be a field and $R = \oplus_{i \geqslant 0} R_i$ be a finitely generated graded k-algebra with $R_0 = k$ and homogeneous maximal ideal $\mathfrak{m} = R_+ = \oplus_{i \geqslant 1} R_i$. We saw in Remark 2.22 that there is a graded version of the Auslander–Buchsbaum formula, where we consider the depth of a graded module M to be the largest length of a regular sequence on M consisting of homogeneous elements. We also saw that this graded depth matches the depth of the module localized at the homogeneous maximal ideal $\mathfrak{m}$: $\operatorname{depth}(M) = \operatorname{depth}_{R_\mathfrak{m}}(M_m)$.

Theorem 2.53 tells us that R is Cohen-Macaulay if and only if $R_\mathfrak{m}$ is Cohen-Macaulay. By definition, $R_\mathfrak{m}$ is Cohen-Macaulay if and only if $\operatorname{depth}(R_\mathfrak{m}) = \dim(R_\mathfrak{m}) = \operatorname{height}(\mathfrak{m})$. We conclude that R is Cohen-Macaulay if and only if $\operatorname{depth}(R) = \operatorname{height}(\mathfrak{m})$.

Corollary 2.55. *Let k be a field and $R = \oplus_{i \geqslant 0} R_i$ be a finitely generated graded k-algebra with $R_0 = k$ and homogeneous maximal ideal $\mathfrak{m} = R_+ = \oplus_{i \geqslant 1} R_i$. The following are equivalent:*

1) *R is Cohen-Macaulay.*
2) *$R_\mathfrak{m}$ is Cohen-Macaulay.*
3) *Any homogeneous system of parameters for R forms a regular sequence.*
4) *Some homogeneous system of parameters for R forms a regular sequence.*

Proof. We have 1 $\Rightarrow$ 2 by definition, 2 $\Rightarrow$ 3 is Lemma 2.50, and 3 $\Rightarrow$ 4 is obvious. If some homogeneous system of parameters forms a regular sequence, then the graded depth of R is maximal: $\operatorname{depth}(R) = \operatorname{height}(\mathfrak{m})$. Thus R is Cohen-Macaulay by Remark 2.54. □

Next, we will show that Cohen-Macaulay rings are catenary. Let us recall the definition.

Definition 2.56. A ring R is **catenary** if for all prime ideals P and Q, there exists a saturated chains of prime ideals

$$P = P_0 \subsetneq \cdots \subsetneq P_n = Q$$

and all such chains have the same length.

Theorem 2.57. *Any quotient of a Cohen-Macaulay ring is catenary.*

Proof. Let S be a Cohen-Macaulay ring and $R = S/I$ for some ideal I. Since the primes of R are in bijective inclusion-preserving correspondence with the primes of S which contain J, it suffices to show that S is catenary.

Let $P \subseteq Q$ be primes in S. The localization R_Q is Cohen-Macaulay, and by Theorem 2.42 we know

$$\operatorname{height}(Q/P) = \dim(R_Q/PR_Q) = \operatorname{height}(Q) - \operatorname{height}(P).$$

This equality holds for any two prime ideals $P \subseteq Q$.

Let $P = P_0 \subseteq P_1 \subseteq \cdots \subseteq P_n = Q$ be a saturated chain of prime ideals, so that

$$\operatorname{height}(P_i/P_{i-1}) = 1 \text{ for all } i.$$

Thus

$$\operatorname{height}(P_i) - \operatorname{height}(Q_i) = \operatorname{height}(P_i/P_{i-1}) = 1$$

for $i = 1, \ldots, n$. Summing these up, we obtain $n = \operatorname{height} Q - \operatorname{height}(P)$. In particular, any two saturated chains of primes from P to Q have length $\operatorname{height} Q - \operatorname{height}(P)$. □

Exercises

Exercise 32. Let $(R, \mathfrak{m}, k)$ be a noetherian local ring and let M be any finitely generated module of infinite projective dimension. Show that $\operatorname{depth}(\Omega_{i+1}(M)) \geqslant \operatorname{depth}(R)$ for all $i \geqslant \operatorname{depth}(R)$.

Exercise 33. Let $(R, \mathfrak{m}, k)$ be a noetherian local ring. Show that R is regular if and only if every finitely generated R-module with $\operatorname{depth}(M) = \operatorname{depth}(R)$ is free.

Exercise 34. Let k be any field and $R = k[x^2, xy, y^2]$.

1) Is x^2, xy, y^2 a regular sequence on R?

2) Is R a Cohen-Macaulay ring? Give an answer with proof.

3) Use Macaulay2 to check whether $\mathbb{Q}[x^2, xy, y^2]$ is a Cohen-Macaulay ring.

Exercise 35. Let k be any field. Determine which of the following rings are Cohen-Macaulay, and carefully justify your answers.

1) $A = k[\![x, y, z]\!]/(x + y + z)$.

2) $B = k[\![x, y, z, w]\!]/(x, y) \cap (z, w)$.

3) $C = k[\![x, y, z, w]\!]/(x^2, xy, yz, zw, w^2)$.

Exercise 36. Let $(R, \mathfrak{m})$ be a noetherian local ring and let M be a finitely generated Cohen-Macaulay R-module. Show that for every $P \in \operatorname{Supp}(M)$, $\dim(M_P) = \operatorname{grade}(P, M)$. In particular, M_P is Cohen-Macaulay for all prime ideals P.

Chapter 3

Completion

3.1 Completions via Cauchy sequences

Definition 3.1. Let R be a ring, I an ideal, and M an R-module. For all $x, y \in M$, we set

$$d_I(x,y) = \frac{1}{\inf\{e > 0 \mid x \not\equiv y \pmod{I^e M}\}} = \frac{1}{\inf\{e > 0 \mid x - y \notin I^e M\}},$$

with the convention that $1/\infty = 0$.

This is a measure of how close x and y are in terms of powers of I, scaled in such a way that if $x - y \in I^e M$ for larger e, then their distance is smaller.

Remark 3.2. Note that $d_I(x,y)$ can only take values 0 or $\frac{1}{e}$ for some positive integer e. Thus $d_I(x,y)$ is completely determined by the integers e such that $d_I(x,y) < \frac{1}{e}$. One of the following must hold:

- If $d_I(x,y) < \frac{1}{e}$ for all e, then $d_I(x,y) = 0$.
- Otherwise, the set of e such that $d_I(x,y) < \frac{1}{e}$ is finite, and if e is the largest such value, then $d_I(x,y) = \frac{1}{e+1}$.

Definition 3.3. Let X be a set. A **metric** on X is a function

$$d \colon X \times X \to \mathbb{R}$$

such that for all $x, y, z \in X$, the following hold:

- **Nonnegativity:** $d(x,y) \geqslant 0$.
- **Positivity:** $d(x,y) = 0 \iff x = y$.
- **Symmetry:** $d(x,y) = d(y,x)$.
- **Triangle inequality:** $d(x,z) \leqslant d(x,y) + d(y,z)$.

The pair (X,d) is called a **metric space**.

The function d_I satisfies most of the axioms of a metric.

Remark 3.4. Note that $d_I(x,y) \geqslant 0$ for all x and y, and if $x = y$ then $d_I(x,y) = 0$.

Moreover, if $x, y, z \in M$ are such that

$$d_I(x,z), d_I(y,z) < 1/e$$

then $x - z, y - z \in I^e M$, so

$$x - y = (x - z) - (y - z) \in I^e M$$

and thus $d_I(x,y) < 1/e$. Therefore,

$$d_I(x,y) \leqslant \max\{d_I(x,z), d_I(y,z)\} \leqslant d_I(x,z) + d_I(y,z).$$

However, d_I does not necessarily satisfy positivity, and thus it might not be a metric. More precisely, $x \neq y$ does not always guarantee $d_I(x,y) > 0$:

Example 3.5. Let R and S be rings. Consider the product ring $R \times S$ as a module over itself, and the ideal $I = 0 \times S$. We have $I = I^e$ for all e, and for all elements $r \in R$ and $s, s' \in S$

$$(r,s) - (r,s') = (0, s - s') \in I^e \text{ for all } e.$$

Thus $d_I((r,s),(r,s')) = 0$. In particular, d_I is not a metric.

But d_I is a pseudometric:

Definition 3.6 (Pseudometric)**.** Let X be a set. A **pseudometric** on X is a function

$$d \colon X \times X \to \mathbb{R}_{\geqslant 0}$$

that satisfies the following properties for all $x, y, z \in X$

- **Nonnegativity:** $d(x,y) \geqslant 0$.
- **Reflexivity:** $d(x,x) = 0$.
- **Symmetry:** $d(x,y) = d(y,x)$.
- **Triangle inequality:** $d(x,z) \leqslant d(x,y) + d(y,z)$.

The pair (X,d) is called a **pseudometric space**.

Remark 3.4 shows that d_I is a pseudometric.

Definition 3.7. Let R be a ring, I an ideal, and M an R-module. The **I-adic topology** on M is the topology with open basis

$$\{m + I^a M \mid m \in M, a \geqslant 1\}.$$

Thus this is the topology whose open sets are arbitrary unions of sets of the form $m+I^aM$.

Remark 3.8. The I-adic topology is the topology arising from the pseudometric d_I: two elements are close if they are congruent modulo a large power of I.

We are primarily interested in the case when $M = R$, where the basic open sets are of the form $r + I^n$ for some $r \in R$.

Let us translate some basic topological notions into this topology.

Lemma 3.9. *Let R be ring, I an ideal, and M an R-module. Let $\{m_n\}$ be a sequence of elements in M and $m \in M$.*

1) *In the I-adic topology, $\lim m_n = m$ if and only if for all $e \geqslant 1$, there exists d such that $m_n \equiv m \pmod{I^eM}$ for all $n \geqslant d$.*

2) *In particular, $\lim m_n = 0$ in the I-adic topology if and only if for any $e \geqslant 1$, there is some $d \geqslant 1$ such that for all $n \geqslant d$, $m_n \in I^eM$.*

3) *A sequence $\{m_n\}$ is Cauchy if and only if for any $e \geqslant 1$, there is some $d \geqslant 1$ such that for all $n, n' \geqslant N$, $m_n \equiv m_{n'} \pmod{I^eM}$.*

4) *Let S be a ring, J an ideal of S, and N an S-module. A function $f\colon M \to N$ is continuous with respect to the I-adic and J-adic topologies if for any $e \geqslant 1$, there is some $d \geqslant 1$ such that*

   $$m \equiv m' \pmod{I^dM} \implies f(m) \equiv f(m') \pmod{J^eN} \iff f(m+I^dM) \subseteq f(m)+J^eN.$$

   In particular, if $\phi : R \to S$ is a ring homomorphism and $\phi(I) \subseteq J$, then ϕ is continuous with respect to the I-adic and J-adic topologies, and any R-module homomorphism $\alpha : M \to N$ is continuous.

Definition 3.10. We say that an R-module M is **I-adically complete** if every Cauchy sequence in M has a unique limit. If $(R, \mathfrak{m})$ is a noetherian local ring, we say M is **complete** to mean M is $\mathfrak{m}$-adically complete. In particular, a complete local ring is a local ring that is complete with respect to its maximal ideal.

Lemma 3.11. *Every Artinian local ring $(R, \mathfrak{m})$ is complete.*

Proof. If R is Artinian, then $\mathfrak{m}^t = 0$ for some t. Therefore, any Cauchy sequence $\{r_n\}$ is eventually constant: there is a d such that for all $n, n' \geqslant d$,

$$r_n \equiv r_{n'} \pmod{\mathfrak{m}^t} \quad \text{and thus} \quad r_n = r_{n'}. \qquad \square$$

Example 3.12. Let A be any ring and consider the polynomial ring $R = A[x]$. Then R is not (x)-adically complete, since the sequence

$$\left\{ \sum_{i=0}^{n} x^i \right\}$$

is Cauchy but has no limit: the limit would have arbitrarily large degree.

Exercise 37. Let A be any ring. Show that $A[\![x_1, \dots, x_t]\!]$ is $(x_1, \dots, x_t)$-adically complete.

Remark 3.13. Let I and J be ideals in a ring R. If R is I-adically complete, then the ring R/J is $I(R/J)$-adically complete. In particular, a quotient of a power series ring over a field is a complete local ring.

Exercise 38. Let R be a ring, I an ideal, and M an R-module. Let $\{r_n\}$ and $\{s_n\}$ be sequences in R and $\{m_n\}$ and $\{l_n\}$ be sequences in M. Show the following:

1) If $\{r_n\}$ and $\{s_n\}$ are Cauchy, then so are $\{r_n + s_n\}$ and $\{r_n s_n\}$.

2) If $\lim r_n = 0$ or $\lim s_n = 0$, then $\lim r_n s_n = 0$.

3) If $\{m_n\}$ and $\{l_n\}$ are Cauchy, so is $\{m_n + l_n\}$.

4) If $\{r_n\}$ and $\{m_n\}$ are Cauchy, then so is $\{r_n m_n\}$.

5) If $\lim m_n = 0$ or $\lim r_n = 0$, then $\lim r_n m_n = 0$.

Corollary 3.14. *Given a ring R and an ideal I, the set of Cauchy sequences with pointwise addition and multiplication forms a ring, and the set of sequences that converge to zero forms an ideal in this ring. Moreover, there is a ring homomorphism from R to this ring that sends an element to the associated constant sequence.*

Definition 3.15. Let R be a ring and I an ideal. The **I-adic completion** of R is the ring $\hat{R}^I$ given by the quotient of the ring of Cauchy sequences by the ideal of sequences that converge to zero. When $(R, \mathfrak{m})$ is a local ring, we denote the $\mathfrak{m}$-adic completion of R by $\widehat{R}$.

Remark 3.16. The completion $\hat{R}^I$ is an R-algebra via the map sending $r \in R$ to the class of the constant sequence equal to r.

Remark 3.17. If M is an R-module, the set of Cauchy sequences in M is a module over the ring of Cauchy sequences in R, and the set of sequences converging to zero is a submodule of M. One can check that this induces a well-defined module action of $\hat{R}^I$ on the set of equivalence classes of Cauchy sequences on M modulo sequences converging to zero.

Definition 3.18. Let R be a ring, I an ideal, and M a module. The I-adic completion of M is the $\widehat{R}^I$-module $\widehat{M}^I$ given by the quotient of the ring of Cauchy sequences by sequences converging to zero in M. When $(R, \mathfrak{m})$ is a local ring, we write $\widehat{M}$ for $\widehat{M}^{\mathfrak{m}}$.

Remark 3.19. Let R be a ring, I an ideal, and M a module. Given a Cauchy sequence $\{a_n\}$ in M, by passing to a subsequence, we can assume that for any e, and any $n, m \geqslant e$, $a_n - a_m \in I^e M$; we just keep skipping a few terms. Thus, we can write

$$a_e = a_0 + (a_1 - a_0) + (a_2 - a_1) + \cdots + (a_e - a_{e-1})$$

with $a_e - a_{e-1} \in I^{e-1} M$. In this way, we can represent any element in $\widehat{M}^I$ as a power series with nth term in $I^n M$. Conversely, any power series with nth term in $I^n M$ corresponds to a Cauchy sequence, so it represents an element of the completion.

3.2 Completion via inverse limits

In this section, our main goal is to give a useful alternative description of completion.

Definition 3.20. Let R be a ring and M an R-module. A **filtration** $\mathcal{F} = \{M_n\}_{n\geqslant 0}$ of M is a descending chain of submodules

$$M = M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots.$$

Remark 3.21. A filtration induces an inverse system of R-modules

$$M/M_0 \xleftarrow{\pi_1} M/M_1 \xleftarrow{\pi_2} M/M_2 \leftarrow \cdots$$

where $\pi_n(m+M_n) = m+M_{n-1}$ for all n. These maps are well-defined because $M_n \subseteq M_{n-1}$.

Definition 3.22. Let $\mathcal{F}$ be a filtration of an R-module M. The **completion** $\widehat{M}^{\mathcal{F}}$ of M with respect to $\mathcal{F}$ is defined by

$$\widehat{M}^{\mathcal{F}} := \varprojlim M/M_n.$$

Recall that by Theorem 1.86 from Homological Algebra

$$\begin{aligned}\varprojlim M/M_n &= \{\{m_n + M_n\} \in \prod_{n\geqslant 0} M/M_n \mid \pi_n(m_n + M_n) = m_{n-1} + M_{n-1} \text{ for all } n\} \\ &= \{\{m_n + M_n\} \in \prod_{n\geqslant 0} M/M_n \mid m_n - m_{n-1} \in M_{n-1} \text{ for all } n\}.\end{aligned}$$

For each filtration $\mathcal{F}$, there is a canonical map $\phi_M^{\mathcal{F}} : M \to \widehat{M}^{\mathcal{F}}$ given by

$$\phi_M^{\mathcal{F}}(m) = \{m + M_n\}_{n\geqslant 0}.$$

Definition 3.23. If $\phi_M^{\mathcal{F}}$ is injective, meaning that

$$\bigcap_n M_n = 0,$$

then M is said to be **separated** with respect to $\mathcal{F}$.

Definition 3.24. Let M be an R-module and I an ideal of R. The filtration of M given by

$$\mathcal{F} = \{I^n M\}_{n\geqslant 0}$$

is the **I-adic filtration** of M.

Definition 3.25. Let M be an R-module and I an ideal in R. The **I-adic completion** of M is the completion of M with respect to the I-adic filtration:

$$\widehat{M}^{I} := \varprojlim M/I^n M.$$

When the canonical map $\phi_M^I : M \to \widehat{M}$ is an isomorphism, we say that M is **I-adically complete**.

Remark 3.26. Let R be a ring and $\mathcal{F} = \{I_n\}$ a filtration on R. Since the projection maps $\pi_n \colon R/I_n \to R/I_{n-1}$ are ring homomorphisms, the completion $\widehat{R}^{\mathcal{F}} = \varprojlim R/I_n$ is a (commutative) ring and the canonical map $\phi_R^{\mathcal{F}} : R \to \widehat{R}^{\mathcal{F}}$ is a ring homomorphism. Similarly, for any R-module M and ideal I, $\widehat{M}^I$ is an $\widehat{R}^I$-module. Thus, the I-adic completion is a (covariant) functor from the category of R-modules to the category of $\widehat{R}^I$-modules.

Our two definitions of completion agree:

Theorem 3.27. *Let R be a ring and I an ideal. There is an R-algebra isomorphism*

$$\widehat{R}^I \cong \varprojlim R/I^n = \{(\bar{r}_1, \bar{r}_2, \bar{r}_3, \dots) \in R/I \times R/I^2 \times R/I^3 \times \cdots \mid \bar{r}_{n+1} \equiv \bar{r}_n \pmod{I^n}\}.$$

More generally, for any R-module M,

$$\widehat{M}^I \cong \varprojlim M/I^n M.$$

Proof. The elements of $\varprojlim R/I^n$ are the sequences

$$\{\bar{r}_n\} \in \prod_{n \geqslant 0} R/I^n$$

that are *consistent*: for all n, the image of $\bar{r}_{n+1}$ under the quotient map $R/I^{n+1} \to R/I^n$ is $\bar{r}_n$. Note that the set of consistent sequences forms a subring of $\prod_{n \geqslant 0} R/I^n$.

We claim that there is an isomorphism from $\widehat{R}^I$ to $\varprojlim R/I^n$. To define it, note that for a Cauchy sequence $\{r_n\}$ and any e, the Cauchy condition implies that the value of r_n modulo I^e stabilizes at some point; set $\bar{r}_e$ to be that value. Then $\{\bar{r}_e\}$ satisfies the compatibility condition above. Given Cauchy sequences $\{r_n\}$ and $\{s_n\}$, one can check that the associated sequences respect sum and product. Furthermore, a sequence $\{r_n\}$ converges to zero if and only if the stable value of r_n modulo I^e is zero for all e. Therefore, we have a well-defined injective map

$$\widehat{R}^I \longrightarrow \varprojlim R/I^n.$$

Given any sequence in $\varprojlim R/I^n$, taking arbitrary lifts for the elements to R yields a Cauchy sequence, so this map is also surjective.

While we leave the details to the reader, the same construction works just as well for modules, giving us an isomorphism of $\widehat{R}^I$-modules

$$\widehat{M}^I \cong \varprojlim M/I^n M. \qquad \square$$

Example 3.28. If $R = A[x_1, \dots, x_n]$, then $\widehat{R}^{(\underline{x})} \cong A[\![x_1, \dots, x_n]\!]$.

Example 3.29. Let p be a prime integer. The (p)-adic completion of $\mathbb{Z}_{(p)}$ is known as the ring of p-adic integers $\widehat{\mathbb{Z}_{(p)}}$.

Theorem 3.30. *Let R be a ring, I be a finitely generated ideal, and M be an R-module. The completion $\widehat{M}^I$ is $I\widehat{R}^I$-adically complete.*

3.3 Properties of completion

Definition 3.31. Let $\mathcal{F} = \{A_n\}_{n\geqslant 1}$ and $\mathcal{G} = \{B_n\}_{n\geqslant 1}$ be two filtrations of an R-module M. We say that $\mathcal{F}$ and $\mathcal{G}$ are **cofinal** with each other if for all n there exist k such that $B_{n+k} \subseteq A_n$ and $A_{n+k} \subseteq B_n$.

Exercise 39. Let M be an R-module and $\mathcal{F} = \{M_n\}$ a filtration of M. Show that:

1) If $M_n = 0$ for some n, then $\phi_M^{\mathcal{F}} : M \to \widehat{M}^{\mathcal{F}}$ is an isomorphism.

2) If $\mathcal{G} = \{L_n\}$ is a filtration of M which is cofinal with $\mathcal{F}$, then $\widehat{M}^{\mathcal{F}} \cong \widehat{M}^{\mathcal{G}}$.

Theorem 3.32. *Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be an exact sequence of R-modules and $\mathcal{F} = \{B_n\}$ a filtration of B. Let $\mathcal{F}' = \{f^{-1}(B_n)\}$ and $\mathcal{F}'' = \{g(B_n)\}$ be the induced filtrations of A and C, respectively. Then*

$$0 \longrightarrow \widehat{A}^{\mathcal{F}'} \longrightarrow \widehat{B}^{\mathcal{F}} \longrightarrow \widehat{C}^{\mathcal{F}''} \longrightarrow 0$$

is exact.

Proof. Let M be an R-module and let $\mathcal{G} = \{M_n\}$ be a filtration of M. Let

$$\widetilde{M}^{\mathcal{G}} := \prod_n M/M_n$$

and define $d^{\widetilde{M}} : \widetilde{M}^{\mathcal{G}} \to \widetilde{M}^{\mathcal{G}}$ by

$$d^{\widetilde{M}}\left(\{m_n + M_n\}\right) = \{(m_n - m_{n+1}) + M_n\}.$$

One can check that $d^{\widetilde{M}}$ is a surjective R-module homomorphism and

$$\ker\left(d^{\widetilde{M}}\right) = \widehat{M}^{\mathcal{G}}.$$

We then have the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \widetilde{A}^{\mathcal{F}'} & \xrightarrow{\tilde{f}} & \widetilde{B}^{\mathcal{F}} & \xrightarrow{\tilde{g}} & \widetilde{C}^{\mathcal{F}''} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle d^{\widetilde{A}}} & & \downarrow{\scriptstyle d^{\widetilde{B}}} & & \downarrow{\scriptstyle d^{\widetilde{C}}} & & \\
0 & \longrightarrow & \widetilde{A}^{\mathcal{F}'} & \xrightarrow{\tilde{f}} & \widetilde{B}^{\mathcal{F}} & \xrightarrow{\tilde{g}} & \widetilde{C}^{\mathcal{F}''} & \longrightarrow & 0
\end{array}$$

where

$$\tilde{f}\left(\{a_n + f^{-1}(B_n)\}\right) = \left(\{f(a_n) + B_n\}\right) \quad \text{and} \quad \tilde{g}\left(\{b_n + B_n\}\right) = \{g(b_n) + g(B_n)\}.$$

One can check that the rows are exact component-wise, which we leave as an exercise. The result now follows by the Snake Lemma, which gives us an exact sequence

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \ker\left(d^{\widetilde{A}}\right) & \longrightarrow & \ker\left(d^{\widetilde{B}}\right) & \longrightarrow & \ker\left(d^{\widetilde{C}}\right) & \longrightarrow & \operatorname{coker}\left(d^{\widetilde{A}}\right) \\
 & & \| & & \| & & \| & & \| \\
0 & \longrightarrow & \widehat{A}^{\mathcal{F}'} & \longrightarrow & \widehat{B}^{\mathcal{F}} & \longrightarrow & \widehat{C}^{\mathcal{F}''} & \longrightarrow & 0
\end{array}$$

□

Lemma 3.33. *Let M be an R-module and I an ideal in R.*

1) If N is a submodule of M with $N \supseteq I^k M$ for some $k \geqslant 0$, then $\widehat{M/N}^I \cong M/N$.

2) For all $k \geqslant 0$, we have $\widehat{M}^I / \widehat{I^k M}^I \cong M/I^k M$.

3) The completion $\widehat{M}^I$ is complete with respect to the filtration $\{\widehat{I^n M}^I\}$.

Proof. To show 1, note that if $N \supseteq I^k M$ then $I^k(M/N) = 0$, and thus the result follows by Exercise 39.

To show 2, fix $k \geqslant 0$. We have an exact sequence

$$0 \longrightarrow I^k M \xrightarrow{f} M \xrightarrow{g} M/I^k M \longrightarrow 0.$$

Let $\mathcal{F} = \{I^n M\}$ be the I-adic filtration of M. Then $\mathcal{F}' = \{f^{-1}(I^n M)\} = \{I^k M \cap I^n M\}$ is the filtration that has $I^k M$ for the first n terms, and then becomes $I^n M$ for $n \geqslant k$. This filtration is cofinal with the I-adic filtration of $I^k M$, as they are eventually the same filtration. Therefore,

$$\widehat{I^k M}^{\mathcal{F}'} \cong \widehat{I^k M}^I$$

by Exercise 39. Moreover, by 1 we have $\widehat{M/I^k M} \cong M/I^k M$. Thus, by Theorem 3.32 we get a short exact sequence

$$0 \longrightarrow \widehat{I^k M} \longrightarrow \widehat{M} \longrightarrow M/I^k M \longrightarrow 0$$

and thus the desired isomorphism.

Finally, to show 3 we use 2, which gives us

$$\widehat{M}/\widehat{I^n M} \cong M/I^n M \implies \left(\widehat{\widehat{M}^I}\right)^I = \varprojlim \widehat{M}^I / \widehat{I^n M}^I = \varprojlim M/I^n M = \widehat{M}^I. \qquad \square$$

Our next big goal is to prove the flatness of completion. For that, we need the Artin–Rees Lemma, which is a famous useful fact. To prove it, we will use another useful tool: the Rees algebra.

Definition 3.34. Let $I = (a_1, \dots, a_n)$ be an ideal in R. The **Rees algebra** of I is the graded subring of $R[t]$, where t is an indeterminate, given by

$$R[It] \coloneqq R[at \mid a \in I].$$

Remark 3.35. Consider the grading on $R[t]$ where all the elements of R have degree 0 and t has degree 1. Then the Rees algebra of I, $R[It]$, is a graded subring of $R[t]$, with degree n piece given by $I^n t^n$. Thus,

$$S = R[It] = R \oplus It \oplus I^2 t^2 \oplus \cdots.$$

Notice moreover that if R is noetherian and $I = (a_1, \dots, a_n)$, then

$$R[It] = R[a_1 t, \dots, a_n t]$$

is also a noetherian ring, given that it is a finitely generated R-algebra.

Theorem 3.36 (The Artin-Rees Lemma)**.** *Let R be a noetherian ring, I an ideal, M a finitely generated R-module and N a submodule of M. Then there exists an integer k such that*

$$I^n M \cap N = I^{n-k}(I^k M \cap N)$$

for all $n \geqslant k$.

Proof. Let $I = (a_1, \dots, a_r)$, and consider the Rees algebra $S = R[It]$ of I. Moreover, consider the module $M[t] = M \otimes_R R[t]$, which is the graded $R[t]$-module

$$M \oplus Mt \oplus Mt^2 \oplus \cdots.$$

We can also view $M[t]$ as a graded S-module by restriction of scalars. Note that $N[t]$ is a graded S-submodule of $M[t]$.

Let A be the S-submodule of $M[t]$ generated by the elements of M, which is the degree zero component of $M[t]$:

$$A := R[It]M = M \oplus IMt \oplus I^2Mt^2 \oplus \cdots.$$

This is called the **Rees module** of I and M. Since M is a finitely generated R-module, A is a finitely generated S-module and thus A is noetherian as an S-module. Now consider the S-module $B = A \cap N[t]$. Then

$$B = M \oplus (IM \cap N)t \oplus (I^2M \cap N)t^2 \oplus \cdots.$$

Since A is a noetherian S-module and B is an S-submodule of A, we conclude that B is finitely generated as an S-module. As B is finitely generated and graded, B can be generated over S by finitely many homogeneous elements, say $u_1 t^{m_1}, \dots, u_\ell t^{m_\ell}$. Without loss of generality, we may assume $m_i \leqslant m_{i+1}$ for all i.

Let $k = m_\ell$. We claim that

$$I^n M \cap N = I^{n-k}(I^k M \cap N)$$

for all $n \geqslant k$. On the one hand, $I^{n-k}(I^k M \cap N) \subseteq I^n M \cap N$. Let $u \in I^n M \cap N$ where $n \geqslant k$. Then $ut^n \in B$. Thus, we can write ut^n in terms of the homogeneous generators of B as an S-module. Hence, there exists $s_i t^{n-m_i} \in S$ for $i = 1, \dots, \ell$ such that

$$\begin{aligned} ut^n &= (s_1 t^{n-m_1})(u_1 t^{m_1}) + \cdots + (s_\ell t^{n-m_\ell})(u_\ell t^{m_\ell}) \\ &= (s_1 u_1 + \cdots + s_\ell u_\ell)t^n \end{aligned}$$

Hence, $u = s_1 u_1 + \cdots + s_\ell u_\ell$. Now, as $s_i t^{n-m_i} \in S$, we have $s_i \in I^{n-m_i}$ for each i. Similarly, as $u_i t^{m_i} \in B$, we have $u_i \in I^{m_i} M \cap N$. Therefore,

$$u \in \sum_{i=1}^{\ell} I^{n-m_i}(I^{m_i} M \cap N).$$

But since $I^r(I^p M \cap N) \subseteq I^{r-1}(I^{p+1} M \cap N)$ for all integers r and p, we have

$$I^{n-m_i}(I^{m_i} M \cap N) \subseteq I^{n-m_\ell}(I^{m_\ell} M \cap N) = I^{n-k}(I^k M \cap N).$$

Therefore, $u \in I^{n-k}(I^k M \cap N)$. □

The Artin-Rees Lemma leads to a very useful version of Krull's Intersection Theorem:

Theorem 3.37 (Krull's Intersection Theorem)**.** *Let $(R, \mathfrak{m})$ be a noetherian local ring, M a finitely generated R-module, and I a proper ideal in R. Then*

$$\bigcap_{n \geqslant 0} I^n M = 0.$$

Proof. Set N to be the intersection we want to show is zero. Note that $I^n M \cap N = N$ for all n. Taking k such as in the Artin-Rees Lemma, Theorem 3.36, and setting $n = k+1$, we obtain

$$IN = I(I^k M \cap N) = I^{k+1} M \cap N = N.$$

By NAK, $N = 0$. □

Theorem 3.38. *Let R be a noetherian ring and I an ideal of R. Let $\widehat{(-)}$ denote I-adic completion. Suppose*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is an exact sequence of finitely generated R-modules. Then

$$0 \longrightarrow \widehat{A} \longrightarrow \widehat{B} \longrightarrow \widehat{C} \longrightarrow 0$$

is exact.

Proof. By identifying A with its image in B, we may assume A is a submodule of B. The I-adic filtration of B induces the filtration $\mathcal{F} = \{I^n B \cap A\}$ on A. By Artin-Rees, Theorem 3.36, there exists an integer k such that for all $n \geqslant 0$ we have

$$I^{n+k} B \cap A = I^n (I^k B \cap A) \subseteq I^n A.$$

Moroever,

$$I^n A \subseteq I^n B \cap A.$$

We conclude that $\mathcal{F}$ is cofinal with the I-adic filtration of A. By Exercise 39, $\widehat{M}^{\mathcal{F}} \cong \widehat{M}^I$.

Moreover, $C \cong B/A$, and the I-adic filtration of B induces a filtration on C, which is precisely $\mathcal{G} = \{I^n B + A/A\} = \{I^n \cdot B/A\}$ the I-adic filtration on C. The result now follows by Theorem 3.32. □

Theorem 3.39. *Let R be a noetherian ring and I an ideal of R. For any finitely generated R-module M, $\widehat{R}^I \otimes_R M \cong \widehat{M}^I$.*

Proof. Let $F \cong R^a$ be a finitely generated free R-module. Then

$$\widehat{F} = \varprojlim F/I^n F \cong \bigoplus_{i=1}^{a} \varprojlim R/I^n \cong (\widehat{R})^a \cong \widehat{R}^I \otimes_R R^a \cong \widehat{R}^I \otimes_R F.$$

Now let M be a finitely generated R-module and $F \to G \to M \to 0$ a presentation of M by finitely generated free R-modules. Then we have a commutative diagram

$$\begin{array}{ccccccc} \widehat{F} & \longrightarrow & \widehat{G} & \longrightarrow & \widehat{M} & \longrightarrow & 0 \\ \downarrow\cong & & \downarrow\cong & & & & \\ \widehat{R} \otimes_R F & \longrightarrow & \widehat{R} \otimes_R G & \longrightarrow & \widehat{R} \otimes_R M & \longrightarrow & 0 \end{array}$$

where the top row is exact by Theorem 3.38. By a diagram chase, one can show that there exists a unique map $\psi : \widehat{M} \to \widehat{R} \otimes_R M$ which makes the following diagram commute:

$$\begin{array}{ccccccc} \widehat{F} & \longrightarrow & \widehat{G} & \longrightarrow & \widehat{M} & \longrightarrow & 0 \\ \downarrow\cong & & \downarrow\cong & & \downarrow\psi & & \\ \widehat{R} \otimes_R F & \longrightarrow & \widehat{R} \otimes_R G & \longrightarrow & \widehat{R} \otimes_R M & \longrightarrow & 0 \end{array}$$

By the Five Lemma, Theorem 2.33 from Homological Algebra, ψ is an isomorphism. □

Theorem 3.40. *Let R be a noetherian ring and I an ideal of R. Then $\widehat{R}^I$ is a flat R-module.*

Proof. Consider any inclusion $L \subseteq M$. If L and M are both finitely generated, then $L \otimes_R \widehat{R}^I \cong \widehat{L}^I$ and $M \otimes_R \widehat{R}^I \cong \widehat{M}^I$ by Theorem 3.39, and by Theorem 3.38

$$0 \longrightarrow L \otimes_R \widehat{R}^I \longrightarrow M \otimes_R \widehat{R}^I$$

must also be exact.

Now let M and N be any two R-modules, possibly not finitely generated, and suppose by contradiction that $\hat{R}^I \otimes_R L \to \hat{R}^I \otimes_R M$ is not injective. Any particular element z in the kernel of $\hat{R}^I \otimes_R L \to \hat{R}^I \otimes_R M$ is a linear combination of simple tensors using finitely many generators $S \subseteq L$. More precisely, if $z = r_1 \otimes a_1 + \cdots + r_n \otimes a_n$, consider the submodule $L' = Ra_1 + \cdots + Ra_n$ of L, which is finitely generated. Moreover, the fact that its image in $\hat{R}^I \otimes_R M$ is zero can be expressed using a combination of finitely many relations on the generators of M. Let M' the submodule of M generated by the elements occurring in the relations on generators of M making the element zero. Note that the original inclusion map restricts to an injective map

$$0 \longrightarrow L' \longrightarrow M'.$$

Then z is in the kernel of the corresponding map $\widehat{R}^I \otimes_R L' \to \widehat{R}^I \otimes_R M'$, but now L' and M' are finitely generated, so we know that $\widehat{R}^I \otimes_R L' \to \widehat{R}^I \otimes_R M'$ has to be injective. We conclude that $z = 0$, and $\widehat{R}^I$ is a flat R-module. □

Exercise 40. Let R be a noetherian ring, I be an ideal in R, and let $\widehat{(-)}$ the I-adic completion functor. Show that $\widehat{R^n} \cong \widehat{R}^n$ for all $n \geqslant 1$.

Theorem 3.41. *Let R be a noetherian ring and I and J be ideals in R. Then*

$$\widehat{J}^{I} = J\widehat{R}^{I} \quad \text{and} \quad \widehat{R/J} \cong \widehat{R}/J\widehat{R}.$$

Moreover, for any finitely generated R-module M, we have $J\widehat{M} = \widehat{J}\,\widehat{M} = \widehat{JM}$.

Proof. Let $\widehat{(-)}$ denote the I-adic completion functor.

By Theorem 3.39, $\widehat{J} \cong J \otimes_R \widehat{R}$, and by Theorem 3.40, $\widehat{R}$ is a flat R-module, so we get an inclusion

$$0 \longrightarrow \widehat{J} \cong J \otimes_R \widehat{R} \longrightarrow R \otimes_R \widehat{R}.$$

Composing with the canonical isomorphism $R \otimes_R \widehat{R} \cong \widehat{R}$ induced by $r \otimes s \mapsto rs$, we get an exact sequence

$$0 \longrightarrow J \otimes_R \widehat{R} \longrightarrow R \otimes_R \widehat{R} \xrightarrow{\cong} \widehat{R}$$

with image $J\widehat{R}$. But this is the inclusion of $\widehat{J}$ in $\widehat{R}$, so we conclude that $\widehat{J} = J\widehat{R}$.

On the other hand, by Theorem 3.38 the short exact sequence

$$0 \longrightarrow J \longrightarrow R \longrightarrow R/J \longrightarrow 0$$

yields a short exact sequence

$$0 \longrightarrow \widehat{J} \longrightarrow \widehat{R} \longrightarrow \widehat{R/J} \longrightarrow 0$$

and thus

$$\widehat{R/J} \cong \widehat{R}/\widehat{J} = \widehat{R}/J\widehat{R}.$$

Finally, fix any finitely generated R-module M. We now have

$$\widehat{J}\,\widehat{M} = (J\widehat{R})\widehat{M} = J\widehat{M}.$$

Moreover,

$$\begin{aligned}
\widehat{R}/\widehat{J} \otimes_{\widehat{R}} \widehat{M} &\cong (R/J \otimes_R \widehat{R}) \otimes_{\widehat{R}} (\widehat{R} \otimes_R M) \quad \text{by Theorem 3.39} \\
&\cong \widehat{R} \otimes_R R/J \otimes_R M \\
&\cong \widehat{R} \otimes_R M/JM \\
&\cong \widehat{M}/\widehat{JM}.
\end{aligned}$$

Since tensoring is always right exact, applying $- \otimes_{\widehat{R}} \widehat{M}$ to the short exact sequence

$$0 \longrightarrow \widehat{J} \longrightarrow \widehat{R} \longrightarrow \widehat{R}/\widehat{J} \longrightarrow 0$$

gives us a right exact sequence

$$\widehat{J} \otimes_{\widehat{R}} \widehat{M} \longrightarrow \widehat{M} \longrightarrow \widehat{M}/\widehat{JM} \longrightarrow 0$$

where the image of the first map is $\widehat{J}\,\widehat{M} \subseteq \widehat{M}$ and the kernel of the second map is $\widehat{JM}$. Thus $\widehat{J}\,\widehat{M} = \widehat{JM}$. □

Theorem 3.42. *Let R be a noetherian ring and $I = (a_1, \dots, a_n)$ an ideal in R. The I-adic completion of R is*

$$\widehat{R}^I \cong R[\![x_1, \dots, x_n]\!]/(x_1 - a_1, \dots, x_n - a_n).$$

In particular, $\widehat{R}^I$ is a noetherian ring.

Proof. Let $S = R[x_1, \dots, x_n]$, $J = (x_1, \dots, x_n)$, and $L = (x_1 - a_1, \dots, x_n - a_n)$. Note that $R \cong S/L$, and the J-adic topology on $R \cong S/L$ considered as an S-module is the same as the I-adic topology on R, as $JR = I$. By Theorem 3.38, the short exact sequence of S-modules

$$0 \longrightarrow L \longrightarrow S \longrightarrow R \longrightarrow 0$$

yields a short exact sequence

$$0 \longrightarrow \widehat{L}^{(\underline{x})} \longrightarrow \widehat{S}^{(\underline{x})} \longrightarrow \widehat{R}^{(\underline{x})} \longrightarrow 0$$

and thus by Theorem 3.41

$$\widehat{R}^I \cong \widehat{S}^{(\underline{x})}/\widehat{L}^{(\underline{x})} = \widehat{S}^{(\underline{x})}/L\widehat{S}^{(\underline{x})} = R[\![x_1, \dots, x_n]\!]/(x_1 - a_1, \dots, x_n - a_n). \qquad \square$$

Rings of power series in finitely many variables over noetherian rings are noetherian, and quotients of neotherian rings are noetherian, so $\widehat{R}^I$ is noetherian.

Theorem 3.43. *Let $R = A[x_1, \dots, x_n]/I$ with A noetherian. Then*

$$\widehat{R}^{(\underline{x})} \cong A[\![x_1, \dots, x_n]\!]/IA[\![x_1, \dots, x_n]\!].$$

Proof. Set $S = A[x_1, \dots, x_n]$. The completion of R as a ring is the same as its completion as an S-module, as Cauchy and convergent sequences are the same under each perspective, so $\widehat{R}^{(\underline{x})}$ can be thought of as the completion of the S-module S/I. Since $\widehat{S}^{(\underline{x})} \cong A[\![x_1, \dots, x_n]\!]$, the short exact sequence of finitely generated S-modules

$$0 \longrightarrow I \longrightarrow S \longrightarrow R \longrightarrow 0$$

leads to the short exact sequence

$$0 \longrightarrow \hat{I}^{(\underline{x})} \longrightarrow \widehat{S}^{(\underline{x})} \longrightarrow \widehat{R}^{(\underline{x})} \longrightarrow 0.$$

By Theorem 3.41,

$$\widehat{R}^{(\underline{x})} \cong \widehat{S}^{(\underline{x})}/I\widehat{S}^{(\underline{x})} = A[\![x_1, \dots, x_n]\!]/IA[\![x_1, \dots, x_n]\!]. \qquad \square$$

Now that we have seen many good properties of completion, we turn to some of the pitfalls.

While we have shown that $\widehat{R}^I$ is a flat R-module and that the completion functor agrees with tensoring with $\widehat{R}^I$ *for finitely generated modules*, the I-adic completion functor need not be left exact, even over noetherian rings.

Example 3.44. Consider the inclusion of $\mathbb{Z}$-modules $\mathbb{Z} \subseteq \mathbb{Q}$ and the ideal $I = (2)$ in $\mathbb{Z}$. By Lemma 3.33

$$\widehat{\mathbb{Z}}^I / \hat{I}^I \cong \mathbb{Z}/I \neq 0$$

so in particular

$$\widehat{\mathbb{Z}}^I \neq 0.$$

On the other hand, $I^n\mathbb{Q} = \mathbb{Q}$ for all n, so

$$\widehat{\mathbb{Q}}^I = \varprojlim \mathbb{Q}/I^n\mathbb{Q} = 0.$$

Thus

$$0 \longrightarrow \widehat{\mathbb{Z}}^I \longrightarrow \widehat{\mathbb{Q}}^I$$

is not exact, and the completion functor $\widehat{(-)}^I$ is not left exact.

One strange thing about completions is that prime ideals do not necessarily stay prime when passing to the completion. In particular, the completion of a domain is not always a domain. The exercise below contains an example.

Exercise 41. Let $k = \mathbb{C}$ and consider the $R = k[x, y]/(y^2 - x^2 - x^3)$.

1) Show that R is a domain.

2) For any rational number n and any nonnegative integer s, let

$$\binom{n}{s} = \frac{n(n-1)\cdots(n-s+1)}{s!}$$

and consider the power series

$$f = \sum_{s \geqslant 0} \binom{\frac{1}{2}}{s} x^s = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \cdots \in k[\![x, y]\!].$$

Show that $f^2 = 1 + x$.

3) Prove that $\widehat{R}^{(x,y)}$ is not a domain.

3.4 Completions of local rings

We now focus on the case of noetherian local rings $(R, \mathfrak{m})$, and their completion with respect to the maximal ideal $\mathfrak{m}$, which we write simply as $\widehat{R}$.

Theorem 3.45. *Let $(R, \mathfrak{m}, k)$ be a noetherian local ring. Then $\widehat{R}$ is a complete local ring with maximal ideal $\widehat{\mathfrak{m}} = \mathfrak{m}\widehat{R}$ and residue field k.*

Proof. First, recall from Theorem 3.41 that $\widehat{\mathfrak{m}} = \mathfrak{m}\widehat{R}$. By Lemma 3.33, $\widehat{R}$ is complete with respect to $\{\widehat{\mathfrak{m}R}\}$, and by Theorem 3.41, $\widehat{\mathfrak{m}R} = \mathfrak{m}\widehat{R} = \widehat{\mathfrak{m}}$. We will show that $\widehat{\mathfrak{m}}$ is the unique maximal ideal of $\widehat{R}$, proving $\widehat{R}$ is a complete local ring.

By Theorem 3.41 and Lemma 3.33,

$$\widehat{R}/\mathfrak{m}\widehat{R} \cong \widehat{R/\mathfrak{m}} \cong R/\mathfrak{m} \cong k$$

is a field, so $\widehat{\mathfrak{m}} = \mathfrak{m}\widehat{R}$ is a maximal ideal of $\widehat{R}$. It remains to show that all the elements not in $\widehat{\mathfrak{m}}$ are units. To show that, we need only to prove that any Cauchy sequence $\{a_n\}$ representing an element in $\widehat{R}$ not in $\mathfrak{m}\widehat{R}$ is a unit. First, note that $\{a_n\}$ is a Cauchy sequence whose stable value modulo $\mathfrak{m}$ is nonzero. By passing to a subsequence, we can assume all a_n are units in R. Now, if $a_n - a_{n'} \in \mathfrak{m}^e$, then

$$a_n^{-1} - a_{n'}^{-1} = \frac{a_{n'} - a_n}{a_n a_{n'}} \in \mathfrak{m}^e.$$

This proves that $\{a_n^{-1}\}$ is a Cauchy sequence, and its class is an inverse for $\{a_n\}$. This shows that $\widehat{R}$ is local with maximal ideal $\mathfrak{m}\widehat{R}$. □

The following useful fact has a somewhat technical proof, which we leave for another day.

Theorem 3.46. *Let $(R, \mathfrak{m})$ be a noetherian local ring. Then $\dim(\widehat{R}) = \dim(R)$. More generally, for any finitely generated R-module, $\dim(M) = \dim(\widehat{M})$.*

Another nice fact about complete local rings is that there is a stronger version of NAK:

Theorem 3.47 (Complete NAK)**.** *Let $(R, \mathfrak{m}, k)$ be a complete local ring, and M be a separated R-module, though not necessarily finitely generated. Given $m_1, \dots, m_t \in M$, we have*

$$M = \sum_i Rm_i \quad \iff \quad M/\mathfrak{m}M = \sum_i k\overline{m_i}.$$

Proof. The implication ($\Rightarrow$) is clear.

($\Leftarrow$) If

$$M/\mathfrak{m}M = \sum_i k\overline{m_i}$$

then

$$M = \sum_i Rm_i + \mathfrak{m}M \quad \text{so} \quad \mathfrak{m}M = \sum_i \mathfrak{m}m_i + \mathfrak{m}^2 M.$$

For all $m \in M$, write

$$\begin{aligned} m &= \sum_i r_{i,0} m_i + t_1, & r_{i,0} \in R,\ t_1 \in \mathfrak{m}M \\ t_1 &= \sum_i r_{i,1} m_i + t_2, & r_{i,1} \in \mathfrak{m},\ t_2 \in \mathfrak{m}^2 M \\ t_2 &= \sum_i r_{i,2} m_i + t_3, & r_{i,2} \in \mathfrak{m}^2,\ t_3 \in \mathfrak{m}^3 M \\ &\vdots & \vdots \end{aligned}$$

For each i, the sequence

$$\{r_{i,0}, r_{i,0} + r_{i,1}, r_{i,0} + r_{i,1} + r_{i,2}, \dots\}$$

is Cauchy, so we obtain elements $r_i = \sum_j r_{i,j} \in R$. Consider the element $\tilde{m} = \sum_i r_i m_i \in M$. To compute $m - \tilde{m}$ modulo $\mathfrak{m}^n M$, we may replace r_i by $\sum_{j=0}^{n-1} r_{i,j}$, and we see the difference is congruent to t_n, which is hence congruent to zero. Thus $m - \tilde{m} \in \mathfrak{m}^n M$ for each n. Thus, $m = \tilde{m}$, so $M = \sum_i Rm_i$. □

As a consequence of Theorem 3.45, the canonical map $R \to \widehat{R}$ is a local homomorphism.

Definition 3.48. Let $(R, \mathfrak{m})$ and $(S, \mathfrak{n})$ be local rings. A ring homomorphism $\varphi\colon R \to S$ is a **local map** if $\varphi(\mathfrak{m}) \subseteq \mathfrak{n}$.

Local homomorphisms have nice properties.

Definition 3.49. An R-module F is **faithfully flat** if F is a flat R-module and $F \otimes_R M \neq 0$ for every nonzero R-module M. A homomorphism of rings $\varphi\colon R \to S$ is faithfully flat if S is a faithfully flat module under the R-module structure induced by φ.

Exercise 42. Give an example of a module that is flat but not faithfully flat.

Exercise 43. Show that the following are equivalent:

1) F is faithfully flat.
2) F is flat and for every proper ideal I, $IF \neq F$.
3) F is flat and for every maximal ideal $\mathfrak{m}$, $\mathfrak{m}F \neq F$.
4) Any complex of R-module homomorphisms $A \xrightarrow{f} B \xrightarrow{g} C$ is exact if and only if

$$F \otimes_R A \xrightarrow{1 \otimes f} F \otimes_R B \xrightarrow{1 \otimes g} F \otimes_R C$$

is exact.

Exercise 44. Show that any flat local homomorphism must be faithfully flat.

Corollary 3.50. *For all noetherian local rings $(R, \mathfrak{m})$, the completion $\widehat{R}$ is a faithfully flat R-module.*

Theorem 3.51. *Let $(R, \mathfrak{m})$ be a noetherian local ring. Then the following hold for all n:*

1) $\mathfrak{m}^n\widehat{R} = (\widehat{\mathfrak{m}})^n = \widehat{\mathfrak{m}^n}$.

2) $\widehat{R}/\widehat{\mathfrak{m}}^n \cong R/\mathfrak{m}^n$.

3) $\widehat{\mathfrak{m}}^n/\widehat{\mathfrak{m}}^{n+1} \cong \mathfrak{m}^n/\mathfrak{m}^{n+1}$.

Proof.

1) Follows from induction and Theorem 3.41 with $J = \mathfrak{m}$ and $M = \mathfrak{m}^{n-1}$.
2) We have
$$\begin{aligned} \widehat{R}/\widehat{\mathfrak{m}}^n &= \widehat{R}/\widehat{\mathfrak{m}^n} && \text{by 1)} \\ &\cong \widehat{R/\mathfrak{m}^n} && \text{by Theorem 3.41} \\ &\cong R/\mathfrak{m}^n && \text{by Lemma 3.33.} \end{aligned}$$
3) Note that
$$\begin{aligned} \widehat{\mathfrak{m}}^n/\widehat{\mathfrak{m}}^{n+1} &\cong \widehat{\mathfrak{m}^n}/\widehat{\mathfrak{m}^{n+1}} && \text{by 1)} \\ &\cong \mathfrak{m}^n\widehat{/\mathfrak{m}^{n+1}} && \text{by Theorem 3.41} \\ &\cong \mathfrak{m}^n/\mathfrak{m}^{n+1} && \text{by Lemma 3.33.} \quad \square \end{aligned}$$

Theorem 3.52. *Let $(R, \mathfrak{m}, k)$ be a noetherian local ring and M a finitely generated R-module. The completion functor takes a minimal free R-resolution of the finitely generated R-module M to a minimal free $\widehat{R}$-resolution of $\widehat{M}$. In particular,*

$$\beta_i^R(M) = \beta_i^{\widehat{R}}(\widehat{M}) \quad \textit{and} \quad \operatorname{pdim}_R(M) = \operatorname{pdim}_{\widehat{R}}(\widehat{M}).$$

Proof. Consider a minimal free resolution F of M, say

$$\cdots \longrightarrow R^{\beta_2} \longrightarrow R^{\beta_1} \longrightarrow R^{\beta_0} \longrightarrow M \longrightarrow 0$$

involves only finitely generated R-modules. By Theorem 3.39 and Theorem 3.40, tensoring with $\widehat{R}$ yields another exact sequence, with

$$R^n \otimes_R \widehat{R} \cong \widehat{R^n} \cong \widehat{R}^n \quad \text{and } M \otimes_R \widehat{R} \cong \widehat{M}.$$

Thus we get a free $\widehat{R}$-resolution $\widehat{F}$

$$\cdots \longrightarrow \widehat{R}^{\beta_2} \longrightarrow \widehat{R}^{\beta_1} \longrightarrow \widehat{R}^{\beta_0} \longrightarrow \widehat{M} \longrightarrow 0$$

for $\widehat{M}$. The differential of F satisfies $\partial(F) \subseteq \mathfrak{m}F$, and thus the differential of $\widehat{F}$ satisfies $\partial(F) \subseteq \widehat{\mathfrak{m}}\widehat{F}$. Since $\widehat{m}$ is the maximal ideal of $\widehat{R}$, we conclude that $\widehat{F}$ is a minimal free resolution for $\widehat{M}$. $\square$

Exercise 45. Let $(R, \mathfrak{m}, k)$ be a noetherian local ring and let M and N be finitely generated R-modules. Show that

$$\operatorname{Tor}_i^R(M,N) \otimes_R \widehat{R} \cong \operatorname{Tor}_i^{\widehat{R}}(\widehat{M},\widehat{N}) \quad \text{and} \quad \operatorname{Ext}^i_R(M,N) \otimes_R \widehat{R} \cong \operatorname{Ext}^i_{\widehat{R}}(\widehat{M},\widehat{N}).$$

For what follows, we need an elementary fact about finite length modules. First, we recall the definition.

Definition 3.53. A module M has **finite length** if it has a filtration of the form

$$0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_n = M$$

with M_{i+1}/M_i simple for each i; such a filtration is called a **composition series** of **length** n. We say a composition series is **strict** if $M_i \neq M_{i+1}$ for all i. Two composition series are **equivalent** if the collections of composition factors M_{i+1}/M_i are the same up to reordering. The **length** of a finite length module M, denoted $\ell(M)$, is the minimum of the lengths of a composition series of M. If M has does not have finite length, we say that M has infinite length, or $\ell(M) = \infty$.

Exercise 46. Let M be an R-module of finite length $\ell(M)$. Show that

$$\ell_R(M) = \ell_{\widehat{R}}(\widehat{M}).$$

We will now prove that several of our favorite invariants are preserved by completion. Note that we have already shown that betti numbers and projective dimension are preserved by completion, but we nevertheless give a new proof below.

Theorem 3.54. *Let $(R, \mathfrak{m}, k)$ be a noetherian local ring and M a finitely generated R-module.*

1) $\beta_i^R(M) = \beta_i^{\widehat{R}}(\widehat{M})$ *for all* i.

2) $\operatorname{pdim}_R(M) = \operatorname{pdim}_{\widehat{R}}(\widehat{M})$.

3) $\operatorname{depth}_R(M) = \operatorname{depth}_{\widehat{R}}(\widehat{M})$.

4) $\mu_R(M) = \mu_{\widehat{R}}(\widehat{M})$.

Proof. If L is a finitely generated R-module killed by $\mathfrak{m}$, then it has finite length, which matches its dimension as a k-vector space. Notice moreover that R and $\widehat{R}$ have the same residue field, as stated in Theorem 3.45. Thus By Exercise 46,

$$\dim_k(L) = \ell_R(L) = \ell_{\widehat{R}}(\widehat{L}) = \dim_k(\widehat{L}) = \dim_k(L \otimes_R \widehat{R}).$$

Given a finitely generated R-module M, we will apply this to the following k-vector spaces:

$$\operatorname{Tor}_i^R(k,M), \quad \operatorname{Ext}^i_R(k,M), \quad \text{and} \quad M/\mathfrak{m}M.$$

For all $i \geqslant 0$, we have the following:

$$\begin{aligned}
\beta_i^R(M) &= \dim_k \left(\operatorname{Tor}_i^R(k, M)\right) && \text{by Exercise 2} \\
&= \dim_k \left(\operatorname{Tor}_i^R(k, M) \otimes_R \widehat{R}\right) \\
&= \dim_k \left(\operatorname{Tor}_i^{\widehat{R}}(k, \widehat{M})\right) && \text{by Exercise 45} \\
&= \beta_i^{\widehat{R}}(\widehat{M}) && \text{by Exercise 2.}
\end{aligned}$$

In particular,

$$\operatorname{pdim}_R(M) = \sup\{i \mid \beta_i^R(M) \neq 0\} = \sup\{i \mid \beta_i^{\widehat{R}}(\widehat{M}) \neq 0\} = \operatorname{pdim}_{\widehat{R}}(\widehat{M}).$$

Similarly, for all $i \geqslant 0$ we have

$$\begin{aligned}
\dim_k \left(\operatorname{Ext}_R^i(k, M)\right) &\text{by Theorem 2.10} \\
&= \dim_k \left(\operatorname{Ext}_R^i(k, M) \otimes_R \widehat{R}\right) \\
&= \dim_k \left(\operatorname{Ext}_{\widehat{R}}^i(k, \widehat{M})\right) && \text{by Exercise 45.}
\end{aligned}$$

By Theorem 2.10,

$$\operatorname{depth}_{\mathfrak{m}}(M) = \min\{i \mid \operatorname{Ext}_R^i(k, M) \neq 0\} = \min\{i \mid \operatorname{Ext}_{\widehat{R}}^i(k, \widehat{M}) \neq 0\} = \operatorname{depth}_{\widehat{\mathfrak{m}}}(\widehat{M}).$$

Finally,

$$\begin{aligned}
\mu_R(M) &= \dim_k(M/mM) && \text{by NAK} \\
&= \dim_k \left(\widehat{M/mM}\right) \\
&= \dim_k \left(\widehat{M}/\widehat{mM}\right) && \text{by Theorem 3.41} \\
&= \dim_k \left(\widehat{M}/\widehat{m}\widehat{M}\right) && \text{by Theorem 3.41} \\
&= \mu_{\widehat{R}}(\widehat{M}) && \text{by NAK.} \quad \square
\end{aligned}$$

We now turn to some of our favorite classes of singularities, and show they are preserved by completion.

Theorem 3.55. *For any noetherian local ring $(R, \mathfrak{m})$, its completion $\widehat{R}$ at $\mathfrak{m}$ is regular if and only if R is regular.*

Proof. By Theorem 3.54,

$$\operatorname{embdim}(\widehat{R}) = \mu(\widehat{\mathfrak{m}}) = \mu(\mathfrak{m}) = \operatorname{embdim}(R).$$

By Theorem 3.46,

$$\dim(R) = \dim(\widehat{R}).$$

Thus

$$\dim(R) = \operatorname{embdim}(R) \iff \dim(\widehat{R}) = \operatorname{embdim}(\widehat{R}).$$

$\square$

We have seen that regular rings are very nice. It turns out that up to completion, *every* noetherian local ring is a quotient of a regular ring. More precisely, every *complete* local ring is a quotient of a regular local ring.

Theorem 3.56 (Cohen)**.** *Every complete noetherian local ring $(R, \mathfrak{m})$ can be written as a quotient of a regular local ring Q, where $Q = V[\![x_1, \dots, x_d]\!]$ with V a field or a complete characteristic zero discrete valuation ring whose maximal ideal is generated by a prime number p.*

This amazing theorem was I. S. Cohen's PhD thesis. If a local ring R is not complete, we can always take its completion, which is now a quotient of a regular local ring. When our local ring R contains a field k, Cohen's Structure Theorem actually says that R is of the form $R = k[\![x_1, \dots, x_d]\!]/I$. In fact, Cohen's theorem says more.

Theorem 3.57 (Cohen's Structure Theorem)**.** *Let $(Q, \mathfrak{m})$ be a complete regular local ring of dimension d.*

1) *If Q contains a field, then $Q \cong k[\![x_1, \dots, x_d]\!]$ for some field k.*
2) *If $\operatorname{char}(Q) = 0$ and $\operatorname{char}(Q/\mathfrak{m}) = p > 0$ and $p \notin \mathfrak{m}^2$, then Q is isomorphic to a formal power series ring $Q \cong V[\![x_2, \dots, x_d]\!]$ in $d-1$ variables over a complete DVR V, and the maximal ideal of V is generated by p.*
3) *If $\operatorname{char}(Q) = 0$ and $\operatorname{char}(Q/\mathfrak{m}) = p > 0$ with $p \in \mathfrak{m}^2$, then Q is isomorphic to a quotient of a formal power series ring in d variables over a complete DVR by a principal ideal.*

Remark 3.58. Complete regular local rings of types 1) and 2) in Cohen's structure theorem are called **unramified**. Those of type 3) are called **ramified**.

Corollary 3.59. *Any complete local ring is catenary.*

Proof. By Theorem 2.29, regular rings are Cohen-Macaulay. By Theorem 2.57, quotients of Cohen-Macaulay rings are catenary. By Cohen's Structure Theorem, any complete local ring is a quotient of a regular local rings, and thus catenary. □

Remark 3.60. Let R be a complete local ring. By Cohen's Structure Theorem, R is a quotient of some regular local ring $(Q, \mathfrak{m})$. But in fact, we can take Q/I to be a **minimal regular presentation** for R, as follows. If $I \not\subseteq \mathfrak{m}^2$, then take a minimal generator $f \in I$ that is not in $\mathfrak{m}^2$. By Exercise 13, $Q/(f)$ is also a regular ring, so we can replace Q by $Q/(f)$ and I by $I/(f)$, which lowers both the embedding dimension of Q and $\mu(I)$. Repeating this process, which must stop as $\operatorname{embdim}(Q)$ and $\mu(I)$ are finite, we end up with a regular local ring $(Q, \mathfrak{m})$ and an ideal $I \subseteq \mathfrak{m}^2$ such that $R \cong Q/I$. Notice that I contains no minimal generators for $\mathfrak{m}$, so

$$\operatorname{embdim}(R) = \mu(\mathfrak{m}/I) = \mu(\mathfrak{m}) = \operatorname{embdim}(Q) = \dim(Q).$$

This explains the word minimal: we found a regular local ring of dimension $\operatorname{embdim}(R)$ surjecting onto R.

Now for any noetherian local ring, by Theorem 3.54

$$\operatorname{embdim}(\widehat{R}) = \mu(\widehat{\mathfrak{m}}) = \mu(\mathfrak{m}) = \operatorname{embdim}(R).$$

Thus there exists a regular local ring $(Q, \mathfrak{m})$ of dimension $\operatorname{embdim}(R)$ such that $\widehat{R} \cong Q/I$ with $I \subseteq \mathfrak{m}^2$. Any such Q and I form a **minimal regular presentation** for R.

Theorem 3.61. *For all noetherian local rings $(R, \mathfrak{m})$, R is Cohen-Macaulay if and only if its completion $\widehat{R}$ at the maximal ideal is Cohen-Macaulay.*

Proof. By Theorem 3.54,

$$\operatorname{depth}(\widehat{R}) = \operatorname{depth}(R).$$

By Theorem 3.46,

$$\dim(R) = \dim(\widehat{R}).$$

Thus

$$\dim(R) = \operatorname{depth}(R) \iff \dim(\widehat{R}) = \operatorname{depth}(\widehat{R}).$$

□

Exercises

Exercise 47. Let $R = \mathbb{C}[\![x, y]\!]$. While R is a complete local ring, in this problem we will prove R_P is not complete for the prime ideal $P = (x)$.

1) Show that $f = e^{\frac{x}{y}}$ is the limit of a Cauchy sequence in R_P, and thus an element in $\widehat{R_P}$.

2) Show that there are no $p, q \in R$ with $q \notin P$ such that $qf = p$.

   Note that this shows that $f \notin R_P$, and thus R_P is not a complete local ring.

Exercise 48. Let $R = \mathbb{R}[x]_{(x^2+1)}$. Give an explicit description of $\widehat{R}$.

Chapter 4

Complete intersections

Complete intersection rings are the closest to regular rings. In many ways, the class of complete intersections displays a sort of rigidity: there is an abundance of homological invariants that are either well-behaved for complete intersections or absolutely wild for all other classes.

In this chapter, we will give a brief introduction to complete intersections, and give one nice characterization of this class in terms of the conormal module. This is only the beginning of a long and beautiful story that you are encouraged to explore further beyond this class.

4.1 More on finite projective dimension vs depth

Lemma 4.1. *Let R be any noetherian ring and I a nonzero ideal in R. If I is a free R-module, then either $I = R$ or $I = (a)$ for a regular element $a \in R$.*

Proof. Since I is finitely generated, we have $I \cong R^n$ for some $n \geqslant 1$. Any distinct nonzero $x, y \in I$, satisfy the relation $xy - yx = 0$, so there are no two linearly independent elements in I. We conclude that $n = 1$ and $I \cong R$. Let $\{a\}$ be a basis for I, and suppose that I is a proper ideal. The map

$$\begin{aligned} R &\xrightarrow{\pi} I \\ r &\longmapsto ra \end{aligned}$$

is an isomorphism, so

$$\operatorname{ann}_R(a) = \ker(\pi) = 0$$

and a is a regular element in R. □

Exercise 49. Let R be any noetherian local ring. Show that any finite exact sequence of finitely generated free R-modules

$$0 \longrightarrow R^{\beta_d} \longrightarrow \cdots \longrightarrow R^{\beta_0} \longrightarrow 0$$

satisfies

$$\sum_{i=0}^{d} (-1)^i \beta_i = 0.$$

The following statement might appear simple, but it has very powerful consequences, as we will soon see.

Theorem 4.2. *Let I be a nonzero ideal in a noetherian local ring R. If I has finite projective dimension, then* $\operatorname{grade}(I) \geqslant 1$*, that is, I contains a regular element of R.*

Proof. Consider any finite (augmented) free resolution of the nonzero ideal I by finitely generated R-modules

$$0 \longrightarrow R^{\beta_d} \longrightarrow R^{\beta_{d-1}} \longrightarrow \cdots \longrightarrow R^{\beta_1} \longrightarrow R^{\beta_0} \longrightarrow I \longrightarrow 0.$$

Since localization is exact, for all prime ideals P we get a finitely generated free resolution for I_P

$$0 \longrightarrow R_P^{\beta_d} \longrightarrow \cdots \longrightarrow R_P^{\beta_0} \longrightarrow I_P \longrightarrow 0.$$

In particular, I_P has finite projective dimension over R_P.

Let P be an associated prime of R. We saw above that I_P has finite projective dimension over R_P. Since P is an associated prime of R, it consists of only zerodivisors, so $\operatorname{depth}(R_P) = 0$. By the Auslander–Buchsbaum formula, Theorem 2.15, we conclude that I_P must be a free module over R_P. By Exercise 49,

$$\operatorname{rank}_{R_P}(I_P) = r := \sum_i (-1)^i \beta_i.$$

In particular, this rank r does not depend on P.

Take any associated prime P of I (as an R-module). The inclusion of R-modules $I \subseteq R$ gives us $\operatorname{Ass}(I) \subseteq \operatorname{Ass}(R)$, so P is an associated prime of R. Since $P \in \operatorname{Ass}(I) \subseteq \operatorname{Supp}(I)$, we conclude that $I_P \neq 0$. Therefore, there exists an associated prime P of R such that $I_P \neq 0$. We saw above that I_P is a free module of rank r, and since $I_P \neq 0$ we conclude that $r \geqslant 1$.

Now suppose that I contains no regular elements, so that I is contained in the union of the associated primes of R. By Prime Avoidance (see Lemma 3.29 from Commutative Algebra I), I must be contained in some associated prime of R, say P. In particular, $I_P \neq R_P$. We have also shown that I_P must be a free module over R_P of rank $r \geqslant 1$, and thus $I_P \neq 0$. By Lemma 4.1, we conclude that I_P is generated by a regular element in R_P. Since $P \in \operatorname{Ass}(R)$, there are no regular elements in R_P, and thus we have reached a contradiction. Therefore, I contains a regular element. □

The projective dimension of an ideal is also connected to its grade.

Exercise 50. Let $(R, \mathfrak{m}, k)$ be a noetherian local ring and I be an ideal of finite projective dimension. Show that $\operatorname{grade}(I) \leqslant \operatorname{pdim}(R/I)$.

Definition 4.3. Let $(R, \mathfrak{m}, k)$ be a noetherian local ring and I be an ideal. We say I is a **perfect ideal** if $\operatorname{pdim}(R/I) = \operatorname{grade}(I)$.

Exercise 51. Let $(R, \mathfrak{m}, k)$ be a Cohen-Macaulay local ring and let I be an ideal of finite projective dimension. Show that I is perfect if and only if R/I is a Cohen-Macaulay R-module.

4.2 Complete intersection rings

Definition 4.4. A noetherian local ring $(R, \mathfrak{m}, k)$ is a **complete intersection** if there exists a regular local ring Q and a regular sequence $\underline{f}$ in Q such that $\widehat{R} \cong Q/(\underline{f})$.

One can show that this is well-defined: given a complete local ring R, if there exists a surjection from a regular local ring $Q \twoheadrightarrow R$ whose kernel is generated by a regular sequence, then the kernel of *any* surjection $Q \twoheadrightarrow R$ is generated by a regular sequence. The details, however, are technical, so we leave them for another day. If we believe this, however, and put it together with the discussion in Remark 3.60, then R is a complete intersection if and only if for any Cohen presentation $\widehat{R} \cong Q/I$ where $(Q, \mathfrak{m})$ is a regular local ring with $I \subseteq \mathfrak{m}^2$, the ideal I is generated by a regular sequence.

Lemma 4.5. *Any regular local ring is a complete intersection.*

Proof. The completion of a regular local ring is regular, by Theorem 3.55. □

Lemma 4.6. *Complete intersection rings are Cohen-Macaulay.*

Proof. Let R be a complete intersection. Its completion $\widehat{R}$ is a quotient of a regular ring by a regular sequence, so it must be Cohen-Macaulay by Theorem 2.33. By Theorem 3.61, R itself must be Cohen-Macaulay. □

Exercise 52. Show that if Q is a regular local ring and $\underline{f}$ is a regular sequence, then $Q/(f)$ is a complete intersection.

Hypersurfaces are the simplest nonregular examples of complete intersections.

Definition 4.7. A **hypersurface** is a quotient ring $R \cong Q/(f)$ where Q is a regular local ring and $f \in \mathfrak{m}^2$.

Remark 4.8. Note that our definition of hypersurface is equivalent to $R \cong Q/(f)$ for a regular local ring Q and a regular element f such that R is not regular.

One way we can make the definition of complete intersection more intrinsic to R is to rely on a koszul complex on a minimal generating set for the maximal ideal of R. Thus we can define the following:

Definition 4.9. Let R be a noetherian local ring, and let $\underline{x}$ be any minimal generating set for the maximal ideal of R. We set

$$\mathrm{H}_i(K^R) := \mathrm{H}_i(\mathrm{kos}(\underline{x}; R))$$

to be the koszul homology on $\underline{x}$, and call it the **koszul homology of** R.

Remark 4.10. By Exercise 10, any two minimal generating sets for the maximal ideal of R lead to isomorphic koszul complexes. In particular, the koszul homology on a minimal generating set for the maximal ideal is independent on the choice of generating set, and $\mathrm{H}_i(K^R)$ does not depend on a choice of minimal generating set for the maximal ideal of R.

Theorem 4.11. *Let R be a noetherian local ring. Then for all i,*

$$\mu(\mathrm{H}_i(K^R)) = \mu(\mathrm{H}_i(K^{\widehat{R}})).$$

Proof. Fix a minimal generating set $\underline{x}$ for the maximal ideal of R, so that $\mathrm{H}_1(R) = \mathrm{H}_1(\underline{x})$, and write $\hat{\underline{x}}$ for the image of $\underline{x}$ in $\widehat{R}$. By Theorem 3.54, $\hat{\underline{x}}$ is a minimal generating set for the maximal ideal $\widehat{m}$ of $\widehat{R}$. Note that

$$\mathrm{kos}(\underline{x}; R) \otimes_R \widehat{R} = \mathrm{kos}(\underline{x}; \widehat{R}) = \mathrm{kos}(\hat{\underline{x}}; \widehat{R}).$$

By Theorem 3.40, $\widehat{R}$ is a flat R-module, and thus tensoring with $\widehat{R}$ commutes with taking homology. In particular,

$$\mathrm{H}_i(K^R) \otimes_R \widehat{R} = \mathrm{H}_i(\mathrm{kos}(\underline{x})) \otimes_R \widehat{R} = \mathrm{H}_i(\mathrm{kos}(\underline{x}) \otimes_R \widehat{R}) = \mathrm{H}_i(\mathrm{kos}(\underline{x}; \widehat{R})) = \mathrm{H}_i(K^{\widehat{R}}).$$

By Theorem 1.35, $M = \mathrm{H}_i(K^R)$ is a finitely generated R-module, and thus by Theorem 3.39,

$$\widehat{\mathrm{H}_i(K^R)} \cong \mathrm{H}_i(K^R) \otimes_R \widehat{R} = \mathrm{H}_i(K^{\widehat{R}}).$$

By Theorem 3.54 we conclude that

$$\mu\left(\mathrm{H}_i(K^R)\right) = \mu\left(\widehat{\mathrm{H}_i(K^R)}\right) = \mu\left(\mathrm{H}_i(\widehat{R})\right). \qquad \square$$

Theorem 4.12. *Let R be a noetherian local ring. Then given any minimal regular presentation for R, say $\widehat{R} \cong Q/I$ with $(Q, \mathfrak{m}, k)$ a regular ring and $I \subseteq \mathfrak{m}^2$,*

$$\mu(\mathrm{H}_1(K^R)) = \mu(I).$$

Proof. By Theorem 4.11, we can reduce to the case when R is complete, so that $R \cong Q/I$. Recall that $\dim(Q) = \mathrm{embdim}(Q) = \mathrm{embdim}(R)$.

Fix a minimal generating set $\underline{y}$ for the maximal ideal of Q. Since Q is a regular ring, $\underline{y}$ is a regular sequence, by Theorem 1.71, and thus by Corollary 1.47 $\mathrm{kos}(\underline{y})$ is a minimal resolution for k over Q. Thus

$$\mathrm{Tor}_1^Q(R, k) = \mathrm{H}_1(\mathrm{kos}(\underline{y}) \otimes_Q R).$$

On the other hand, the image $\underline{x}$ of $\underline{y}$ in R forms a minimal generating set for the maximal ideal of R, so

$$\mathrm{kos}(\underline{y}; Q) \otimes_Q R = \mathrm{kos}(\underline{x}; R) = K^R$$

and

$$\mathrm{Tor}_1^Q(R, k) = \mathrm{H}_1(\mathrm{kos}(\underline{y}; Q) \otimes_Q R) = \mathrm{H}_1(K^R).$$

On the other hand, the short exact sequence

$$0 \longrightarrow I \longrightarrow Q \longrightarrow R \longrightarrow 0$$

yields the following long exact sequence on $\operatorname{Tor}_*(-,k)$:

$$0 = \operatorname{Tor}_1^Q(Q,k) \longrightarrow \operatorname{Tor}_1^Q(R,k) \longrightarrow I \otimes_Q k \longrightarrow Q \otimes_Q k \longrightarrow R \otimes_Q k \longrightarrow 0.$$

Then $Q \otimes_Q k$ and $R \otimes_Q k$ are both naturally isomorphic to k, and the map $Q \otimes_Q k \longrightarrow R \otimes_Q k$ induced by the quotient map $Q \twoheadrightarrow R$ is the identity on k. Thus our exact sequence is

$$0 \longrightarrow \operatorname{Tor}_1^Q(R,k) \longrightarrow I \otimes_Q k \longrightarrow Q \otimes_Q k \xrightarrow{=} k \longrightarrow 0.$$

and we conclude that

$$\operatorname{Tor}_1^Q(R,k) \cong I \otimes_Q k \cong I/\mathfrak{m}I.$$

We saw above that $\operatorname{Tor}_1^Q(R,k) = \mathrm{H}_1(K^R)$, so we conclude that $\mathrm{H}_1(K^R) = I/\mathfrak{m}I$, and thus $\mu(\mathrm{H}_1(K^R)) = \mu(I)$. □

Theorem 4.13. *A noetherian local ring R is a complete intersection if and only if*

$$\mu(\mathrm{H}_1(K^R)) = \operatorname{embdim}(R) - \dim(R).$$

Proof. Let $\widehat{R} \cong Q/I$ be a minimal regular presentation for I, meaning that $(Q,\mathfrak{m})$ is a regular local ring and $I \subseteq \mathfrak{m}^2$. In particular, $\dim(Q) = \operatorname{embdim}(Q) = \operatorname{embdim}(R)$. Since Q is a Cohen-Macaulay ring, by Theorem 2.43

$$\operatorname{height}(I) = \dim(Q) - \dim(Q/I) = \operatorname{embdim}(R) - \dim(R).$$

On the other hand, by Corollary 2.47

$$I \text{ is generated by a regular sequence} \iff \mu(I) = \operatorname{height}(I).$$

By Theorem 4.12, $\mu(I) = \mu(\mathrm{H}_1(K^R))$. Thus

$$I \text{ is generated by a regular sequence} \iff \mu(\mathrm{H}_1(K^R)) = \operatorname{embdim}(R) - \dim(R). \qquad \square$$

This settles the question of giving an intrinsic definition of complete intersections. In a different direction, the following is also a natural question:

Problem 2 (Localization Problem for complete intersections)**.** Let $(R,\mathfrak{m})$ be a noetherian local ring. If R is a complete intersection, is R_P a complete intersection for all primes P?

The answer is easy when R itself is complete:

Exercise 53. Let $(R,\mathfrak{m})$ be a complete local ring, and P a prime in R. Show that if R is a complete intersection, then so is R_P.

In the general case, the challenge is that localization does not commute with taking completion, making the problem quite technical. This was first resolved by Avramov [Avr75] in 1975. Other proofs can be found in the PhD theses of Josh Pollitz [Pol19] and Ryan Watson. The solution to the Localization Problem for regular rings by Auslander, Buchsbaum, and Serre was structural in nature: it involved a homological characterization of regular local rings. Pollitz's proof that complete intersections localize is also structural in natural, and it involves a *homotopical* characterization of complete intersections.

Definition 4.14. A noetherian ring R is a local complete intersection if R_P is a complete intersection for all primes P.

Theorem 4.13 gave us a characterization of complete intersections via the first Koszul homology of R. Another module that is closely tied to the complete intersection property is the conormal module.

Definition 4.15. The **conormal module** of a surjective ring homomorphism $\varphi\colon Q \twoheadrightarrow R$ with kernel I is the module I/I^2.

Remark 4.16. Note that I/I^2 is not only a Q-module but also a module over $R = Q/I$.

Theorem 4.17 (Vasconcelos, 1977 [Vas78], Ferrand, 1977 [Fer67])**.** *Let I be a nonzero ideal of finite projective dimension in a noetherian local ring $(Q, \mathfrak{m}, k)$. Set $R = Q/I$. If I/I^2 is a free R-module, then I is generated by a regular sequence.*

Proof. By Theorem 4.2, I must contain a regular element, and thus I is not contained in the union of the associated primes of R. Moreover, $I \not\subseteq \mathfrak{m}I$ by NAK (see Theorem 5.34 from Commutative Algebra I). By Prime Avoidance, Lemma 3.29 from Commutative Algebra I,

$$I \not\subseteq \mathfrak{m}I \cup \bigcup_{P \in \operatorname{Ass}(R)} P.$$

Thus we can pick a minimal generator $x \in I$ that is also a regular element on R.

We proceed by induction on $\mu(I)$. If $\mu(I) = 1$, then $I = (x)$ is generated by a regular sequence. When $\mu(I) > 1$, note that $I/(x)$ is generated by $\mu(I) - 1$ elements, and since x is regular on R then it is also regular on I. We will prove that

$$\operatorname{pdim}_{Q/(x)}(I/(x)) < \infty \quad \text{and} \quad \frac{I/(x)}{(I/(x))^2} \text{ is a free } R/(x)\text{-module.}$$

Induction will then complete the proof: the induction hypothesis gives us that $I/(x)$ is generated by a regular sequence, which together with x we can lift to a regular sequence generating I.

Since x is regular on R and I, by Exercise 22 we know that

$$\operatorname{Tor}_i^Q(Q/(x), I) = 0 \text{ for all } i \geqslant 1.$$

Given a finite free resolution F for I over Q, note that

$$\mathrm{H}_i(F \otimes_Q Q/(x)) = \mathrm{Tor}_i^Q(Q/(x), I) = 0 \text{ for all } i \geqslant 1$$

and

$$\mathrm{H}_0(F \otimes_Q Q/(x)) = I \otimes_Q Q/(x) = I/(x)I.$$

We conclude that $F \otimes_Q Q/(x)$ is a finite free resolution for $I/(x)I$ over $Q/(x)$. We will use this to prove that $I/(x)$ also has finite projective dimension.

We have

$$\frac{I/I^2}{\mathfrak{m}I/I^2} \cong I/\mathfrak{m}I.$$

Since $x \notin \mathfrak{m}I$, its image in the free module I/I^2 is a basis element, and thus we can extend it to $x, x_2, \dots, x_n$ such that $x + I^2, x_2 + I^2, \dots, x_n + I^2$ is a basis for I/I^2. Set $J = (x_2, \dots, x_n)$.

Set $J = (x_2, \dots, x_n)$. Given any $z \in J \cap (x)$, note that

$$z = ax \text{ and } z = a_2x_2 + \cdots + a_nx_n \text{ for some } a, a_2, \dots, a_n \in Q.$$

But in I/I^2, the elements $x + I^2, x_2 + I^2, \dots, x_n + I^2$ are linearly independent, and thus all the coefficients $a, a_2, \dots, a_n$ are zero in R, so in particular $a \in I$ and $z = ax \in (x)I$. We conclude that $J \cap (x) \subseteq (x)I$. Therefore, by the Third Isomorphism Theorem

$$I/(x) = J + (x)/(x) \cong J/J \cap (x).$$

Moreover, the inclusion $J \subseteq I$ induces a map on the quotients

$$J/J \cap (x) \longrightarrow I/J \cap (x).$$

Composing with the canonical quotient maps $I/J \cap (x) \twoheadrightarrow I/(x)I$ and $I/(x)I \twoheadrightarrow (x)$, we get a homomorphism

$$\begin{aligned} I/(x) \cong J/J \cap (x) &\longrightarrow I/J \cap (x) \longrightarrow I/(x)I \longrightarrow I/(x) \\ x_i + (x) &\longmapsto x_i + (J \cap (x)) \longmapsto x_i + (x)I \longmapsto x_i + I. \end{aligned}$$

Since $I/(x)$ is generated by $x_i + (x)$, this composition is the identity map, giving us a splitting of the surjection $I/(x)I \twoheadrightarrow I/(x)$. We conclude that $I/(x)$ is a direct summand of $I/(x)I$. By Exercise 18, $I/(x)$ also has finite projective dimension over $Q/(x)$.

It remains to show that $\frac{I/(x)}{(I/(x))^2}$ is a free $R/(x)$-module. This quotient

$$\frac{I/(x)}{(I/(x))^2} \cong \frac{I/(x)}{I^2 + (x)/(x)} \cong I/I^2 + (x)$$

is isomorphic to the quotient of I/I^2 by the free submodule generated by $x + I^2$, and thus it is free, with basis given by the images of $x_2, \dots, x_n$. □

Forty years after Ferrand and Vasconcelos' independent proof of Theorem 4.17, Briggs proved a much stronger version of the theorem, which had been conjectured by Vasconcelos back in the 1970s:

Theorem 4.18 (Briggs, 2022 [Bri22]). *Let $(Q, \mathfrak{m}, k)$ be a noetherian local ring and let I be a nonzero ideal of finite projective dimension. If I/I^2 has finite projective dimension over $R = Q/I$, then I is generated by a regular sequence.*

An astonishing consequence of Briggs, Vasconcelos, and Ferrand's work is that if the conormal module has finite projective dimension over R, then in fact it is free. Note that R itself must be Cohen-Macaulay but can have any dimension, so in general there are in fact modules over R that have finite projective dimension but are not free. This fits into a more general theme: there are many interesting properties that display a sort of rigidity with respect to complete intersections, where a ring is either very nice with respect to that property, and thus a complete intersection, or its behavior is completely wild.

Here is another example of such a theorem:

Definition 4.19. Let M be a finitely generated R-module of finite complexity. If M has finite projective dimension, we say M has complexity 0, and write $\operatorname{cx}(M) = 0$. Otherwise, we say M has complexity d, and write $\operatorname{cx}(M) = d$, if $d - 1$ is the smallest degree of a polynomial f with $\beta_i(M) \leqslant f(i)$ for all i.

Remark 4.20. Some authors define the complexity of M to be the least integer d such that $\beta_n(M) \leqslant C \cdot n^{d-1}$ for some constant C and all $n \gg 0$. This has the advantage that it includes complexity 0 as part of the definition. Also, note that this other definition is equivalent to our definition, as only the largest power term of the polynomial really matters, and when $\operatorname{cx}(M) > 0$ we can always change the constant C so that the inequality applies to all (not just large) n.

Example 4.21. Complexity 1 means that M has infinite projective dimension but the Betti numbers are bounded above by a constant. For example, over $R = k[\![x]\!]/(x^2)$, the module k has complexity 1, as $\beta_i(k) = 1$ for all i.

Theorem 4.22 (Gulliksen, 1980 [Gul80]). *Let $(R, \mathfrak{m}, k)$ be a noetherian local ring. The following are equivalent:*

1) R is a complete intersection.

2) Every finitely generated R-module has finite complexity.

3) The residue field k has finite complexity.

Thus if R is not a complete intersection, the betti numbers of the residue field grow exponentially.

Chapter 5

Enough about injectives

We saw in Homological Algebra that every module embeds in some injective module. We will prove something stronger: that every module has a *smallest* injective module it embeds in, called its injective envelope. This will allow us to define *minimal* injective resolutions, and in a story dual to free resolutions, we will prove that minimal injective resolutions are unique in the appropriate sense.

The injective envelope was first defined in 1953, by Eckmann and Schopf ([ES53]), but most of the theory of injective envelopes we will study was established a few years later by Eben Matlis for his PhD thesis ([Mat58]), including the theorem known as Matlis Duality, which we will study later.

Once we have established the existence and uniqueness up to isomorphism of the injective hull, we will prove some basic properties of this object and proceed to study injective modules over noetherian rings. Using the theory of injective envelopes, we will be able to characterize every injective module over a noetherian ring as a direct sum of certain injective envelopes, reducing the study of injective modules over noetherian rings to the study of the injective envelope of R/P, for P a prime ideal in R.

5.1 Essential Extensions

We will often use the following abuse of notation:

Notation 2. Given be an injective module homomorphism $f\colon E \longrightarrow F$ and a submodule G of F, we will sometimes write $G \cap E$ for $G \cap f(E)$.

Definition 5.1 (Essential Extension)**.** Let A and B be R-modules. We say that B is an **essential extension** of A, or that $A \subseteq B$ is an essential extension, if the following hold:

1) There is an injective homomorphism $A \hookrightarrow B$, that is, A embeds into B.

2) For every nonzero submodule M of B, $M \cap A \neq 0$.

We will often abuse notation and assume A is a submodule of B.

Remark 5.2. An inclusion of R-modules $A \subseteq B$ is an essential extension if and only if

$$\{M \mid 0 \neq M \text{ is a submodule of } B \text{ and } M \cap A = 0\} = \emptyset.$$

Remark 5.3. To show that $A \subseteq B$ is an essential extension, it suffices to consider cyclic submodules of B. Indeed, if we show that for all nonzero $b \in B$ we have $Rb \cap A \neq 0$, then for any nonzero submodule $M \subseteq B$ we can take any nonzero $b \in M$, and

$$M \cap A \supseteq Rb \cap A \neq \emptyset.$$

Remark 5.4. Note that if for any nonzero module M, the trivial inclusion $M \subseteq M$ is an essential extension. In particular, every nonzero module has at least one essential extension.

Example 5.5. Over $\mathbb{Z}$, $\mathbb{Q}$ is an essential extension of $\mathbb{Z}$. Indeed, if $M \neq 0$ is any subgroup of $\mathbb{Q}$, there must be some element nonzero $\frac{p}{q} \in M$ with $p, q \in \mathbb{Z}$, and therefore $p = q\,\frac{p}{q} \in M \cap \mathbb{Z}$.

Example 5.6. More generally, given any domain R, and its fraction field $Q = R_{(0)}$, the natural inclusion $R \subseteq Q$ is an essential extension of R-modules: given any nonzero R-submodule M of Q, it contains some nonzero element $\frac{p}{q} \in M$ with $p, q \in R$ and $q \neq 0$, and therefore $p = q\,\frac{p}{q} \in M \cap R$.

Example 5.7. Consider the ring $\mathbb{Z}$, any prime integer p, and the $\mathbb{Z}$-module

$$\mathbb{Z}\left[\frac{1}{p^\infty}\right] = \left\{\frac{a}{p^n} + \mathbb{Z} \in \mathbb{Q}/\mathbb{Z} \mid a \in \mathbb{Z} \text{ and } n \geqslant 1\right\}.$$

For each $n \geqslant 1$, the $\mathbb{Z}$-module homomorphism $\mathbb{Z} \to \mathbb{Z}\left[\frac{1}{p^\infty}\right]$ defined by sending $1 \mapsto \frac{1}{p^n} + \mathbb{Z}$ has kernel (p^n), and thus it induces an injective homomorphism of $\mathbb{Z}$-modules

$$\begin{aligned} \mathbb{Z}/(p^n) &\longrightarrow \mathbb{Z}\left[\tfrac{1}{p^\infty}\right] \\ a + (p^n) &\longmapsto \tfrac{a}{p^n} + \mathbb{Z}. \end{aligned}$$

Let us identify $\mathbb{Z}/(p^n)$ with its image, which is the submodule of $\mathbb{Z}\left[\frac{1}{p^\infty}\right]$ whose elements are of the form $\frac{a}{p^n} + \mathbb{Z}$ for some $a \in \mathbb{Z}$.

We claim this inclusion $\mathbb{Z}/(p^n) \subseteq \mathbb{Z}\left[\frac{1}{p^\infty}\right]$ is an essential extension. To show that, consider any element $\frac{a}{p^m} + \mathbb{Z} \in \mathbb{Z}\left[\frac{1}{p^\infty}\right]$. If $m \leqslant n$, then

$$\frac{a}{p^m} + \mathbb{Z} = \frac{p^{n-m}a}{p^n} + \mathbb{Z} \in \mathbb{Z}/(p^n).$$

If $m \geqslant n$, then

$$p^{m-n}\left(\frac{a}{p^m} + \mathbb{Z}\right) = \frac{a}{p^n} + \mathbb{Z} \in \mathbb{Z}/(p^n).$$

In both cases, $\mathbb{Z} \cdot \left(\frac{a}{p^m} + \mathbb{Z}\right) \cap \mathbb{Z}/(p^n) \neq 0$. This shows that

$$\mathbb{Z}/(p^n) \subseteq \mathbb{Z}\left[\frac{1}{p^\infty}\right].$$

is an essential extension.

Theorem 5.8. *Given R-modules $A \subseteq B \subseteq C$,*

$$C \text{ is an essential extension of } A \iff \begin{array}{c} C \text{ is an an essential extension of } B \\ \text{and} \\ B \text{ is an essential extension of } A. \end{array}$$

Proof. ($\Leftarrow$) Let $M \neq 0$ be a submodule of C. As C is an essential extension of B, we know that $M \cap B \neq 0$. Since $M \cap B$ is a nonzero submodule of B and $A \subseteq B$ is an essential extension, we conclude that

$$M \cap A = (M \cap B) \cap A \neq 0.$$

Therefore, C is an essential extension of A.

($\Rightarrow$) Consider any nonzero submodule M of C. Since $0 \neq M \cap A \subseteq M \cap B$, then $M \cap B \neq 0$. This shows that C is an essential extension of B.

Now consider any nonzero submodule M of B. Since M is also a submodule of C, and $A \subseteq C$ is an essential extension, then we must have $M \cap A \neq 0$. We conclude that M is an essential extension of E. □

Theorem 5.9. *Let R be a noetherian ring. If $A \subseteq B$ is an essential extension of R-modules, then* $\operatorname{Ass}(A) = \operatorname{Ass}(B)$.

Proof. By Lemma 6.28 from Commutative Algebra I, $\operatorname{Ass}(A) \subseteq \operatorname{Ass}(B)$. By Theorem 6.26 from Commutative Algebra I, if $\operatorname{Ass}(B) = \emptyset$, then $B = 0$, and thus $A = 0$, so $\operatorname{Ass}(A) = \emptyset$.

Assume $\operatorname{Ass}(B) \neq \emptyset$. Let $P \in \operatorname{Ass}(B)$, and fix $x \in B$ such that $P = \operatorname{ann}(x)$. Then

$$Rx \cong R/P \implies \operatorname{Ass}(Rx) = \operatorname{Ass}(R/P) = \{P\}.$$

Since $A \subseteq B$ is an essential extension, we know that $Rx \cap A \neq 0$, so again by Theorem 6.26 from Commutative Algebra I, we have $\operatorname{Ass}(Rx \cap A) \neq \emptyset$. Since $Rx \cap A$ is a submodule of Rx and B, by Lemma 6.28 from Commutative Algebra I

$$\emptyset \neq \operatorname{Ass}(Rx \cap A) \subseteq \operatorname{Ass}(Rx) \cap \operatorname{Ass}(B) = \{P\} \cap \operatorname{Ass}(B) \subseteq \{P\}.$$

We conclude that $\{P\} = \operatorname{Ass}(Rx \cap A) \subseteq \operatorname{Ass}(A)$, and in particular $P \in \operatorname{Ass}(A)$. □

The following simple observation will be very useful later.

Lemma 5.10. *Let $A \subsetneq B$ be R-modules. If the short exact sequence induced by the inclusion*

$$0 \longrightarrow A \longrightarrow B \longrightarrow B/A \longrightarrow 0$$

splits, then $A \subseteq B$ is not an essential extension.

Proof. If the short exact sequence splits, then B has a submodule $M \cong B/A \neq 0$ such that $B = A \oplus M$. By definition of internal direct sum, $M \cap A = 0$, and thus $A \subseteq B$ is not an essential extension. □

Corollary 5.11. *Over a field, there are no proper essential extensions of modules.*

Proof. Let k be a field. By Theorem 4.62 from Homological Algebra, every short exact sequence over k splits. By Lemma 5.10, there are no proper essential extensions. □

Remark 5.12. Let R be any noetherian ring and M an R-module. Given a multiplicatively closed subset S of R, by Theorem 6.26 from Commutative Algebra I,

$$M_S \neq 0 \iff \operatorname{Ass}_{R_S}(M_S) \neq \emptyset.$$

By Theorem 6.38 from Commutative Algebra I,

$$\operatorname{Ass}_{R_S}(M_S) = \{P_S \mid P \in \operatorname{Ass}_R(M),\ P \cap S = \emptyset\}.$$

Thus

$$M_S \neq 0 \iff \text{ there exists } P \in \operatorname{Ass}_R(M) \text{ such that } P \cap S = \emptyset.$$

Theorem 5.13. *Let R be a noetherian ring, S a multiplicative set in R, and $A \subseteq B$ an essential extension of R-modules. Then $A_S \subseteq B_S$ is an essential extension of R_S-modules.*

Proof. If $B_S = 0$, then $A_S = 0$ and $A_S \subseteq B_S$ is trivially an essential extension.

Suppose that $B_S \neq 0$ and consider any submodule of B_S, which must be of the form M_S for some submodule M of B. Note that

$$M_S \cap A_S = (M \cap A)_S.$$

so it suffices to show that $(M \cap A)_S \neq 0$. By Remark 5.12, we need to show that there exists an associated prime P of $M \cap A$ such that $P \cap S = \emptyset$.

Since $A \subseteq B$ is an essential extension, so is $A \cap M \subseteq M$ by Theorem 5.8. By Theorem 5.9,

$$\operatorname{Ass}_R(A \cap M) = \operatorname{Ass}_R(M).$$

Since $M_S \neq 0$, by Remark 5.12 there exists $P \in \operatorname{Ass}_R(M) = \operatorname{Ass}_R(A \cap M)$ such that $P \cap S = \emptyset$. We conclude that $(A \cap M)_S \neq 0$, and thus $A_S \subseteq B_S$ is an essential extension.r □

Theorem 5.14. *Let $A \subseteq B$ be R-modules. Then A has an essential extension E with $E \subseteq B$ that is maximal inside B.*

Proof. All we have to check is that Zorn's Lemma applies to the set Γ of essential extensions $A \subseteq E$ such that $E \subseteq B$. Since $A \in \Gamma$, then $\Gamma \neq \emptyset$. If $\{F_\lambda\}_{\lambda \in \Lambda}$ is any family of R-modules in Γ totally ordered by inclusion, then

$$F := \bigcup_{\lambda \in \Lambda} F_\lambda \in \Gamma.$$

The union of a chain of R-modules is always an R-module, and thus F is an R-module. By construction, $F \subseteq B$. We claim that $A \subseteq F$ is an essential extension.

To prove that, consider a nonzero submodule M of F, and any nonzero $x \in M$. There must be some $\lambda \in \Lambda$ for which $x \in F_\lambda$, so that $F_\lambda \cap M \neq 0$. Now $F_\lambda \cap G$ is a nonzero submodule of F_λ, and since $F_\lambda \in \Gamma$ is an essential extension of A, then $(F_\lambda \cap M) \cap A \neq 0$. Then $M \cap A \neq 0$, so that F is indeed an essential extension of A. As $F \subseteq B$, this completes the proof that $F \in \Gamma$. Therefore, the conditions of Zorn's Lemma apply, giving us a maximal essential extension of A inside B. □

5.2 Injective envelopes

We now turn to the connection with injective modules. First, we will see that the localization of an injective module is injective.

Exercise 54. Let R be a noetherian ring and M an R-module. Show that

$$\operatorname{injdim}(M) = \sup\left\{i \colon \operatorname{Ext}^i_R(R/I, M) \neq 0 \mid I \text{ is a proper ideal of } R\right\}.$$

Theorem 5.15. *Let R be a noetherian ring, E an injective R-module, and S a multiplicatively closed subset of R. Then E_S is an injective R_S-module.*

Proof. Since E is an injective R-module, $\operatorname{Ext}^i_R(R/I, E) = 0$ for all ideals I and all $i \geqslant 1$. Note that R/I is a finitely generated R-module. By Theorem 6.35 from Homological Algebra,

$$\operatorname{Ext}^i_{R_S}(R_S/I_S, E_S) \cong \left[\operatorname{Ext}^i_R(R/I, E)\right]_S = 0_S = 0$$

for every $i \geqslant 1$. Every ideal in R_S is of the form I_S. Since, by Exercise 54,

$$\operatorname{injdim}_{R_S}(E_S) = \sup\left\{i \colon \operatorname{Ext}^i_{R_S}(R_S/I_S, E_S) \neq 0, I \text{ an ideal in } R\right\},$$

we must therefore have $\operatorname{injdim}_{R_S}(E_S) = 0$, that is, E_S is an injective R_S-module. □

Theorem 5.16. *A module E is injective if and only if E has no proper essential extensions.*

Proof.

($\Rightarrow$) Let $E \subsetneq F$ for any R-module F. Since E is injective, by Theorem 4.34 from Homological Algebra the short exact sequence

$$0 \longrightarrow E \longrightarrow F \longrightarrow F/E \longrightarrow 0.$$

splits. By Lemma 5.10, $E \subseteq F$ is not an essential extension. We conclude that E has no proper essential extensions.

($\Leftarrow$) Suppose E has no proper essential extensions, and fix an injective module F such that $E \subseteq F$, which exists by Theorem 4.33 from Homological Algebra. If $E \neq F$, then by assumption F cannot be an essential extension of E, so that

$$\Gamma = \{M \mid 0 \neq M \text{ is a submodule of } F \text{ and } M \cap E = 0\} \neq \emptyset,$$

as noted in Remark 5.2. If $\{F_\lambda\}_\lambda$ is a family of R-modules in Γ totally ordered by inclusion, the union

$$U := \bigcup_{\lambda \in \Lambda} F_\lambda$$

is also a submodule of F. Also, for every $x \in U$, we must have $x \in F_\lambda$ for some $\lambda \in \Lambda$, and therefore either $x = 0$ or $x \notin E$. This means that $U \cap E = 0$, and thus $U \in \Gamma$. This guarantees that Zorn's Lemma applies to Γ. Therefore, there exists a maximal element K in

Γ, that is, there exists a nonzero submodule $K \subseteq F$ with $K \cap E = 0$ that is maximal with respect to this property. Notice that $K \neq F$, since $F \cap E = E \neq 0$.

Consider the composition

$$\begin{aligned} \varphi \colon E &\xhookrightarrow{\ i\ } F \xrightarrow{\ \pi\ } F/K \\ e &\longmapsto e \longmapsto e + K \end{aligned}$$

where i is the inclusion map and π the canonical projection. We will prove that F/K is an essential extension of E. Since i is injective, $E = i(E)$, $\ker(\pi) = F$, and $E \cap F = 0$, we conclude that φ is injective, so E embeds into F/K.

Now consider a nonzero submodule of F/K, which must be of the form H/K for some submodule H of F with $K \subsetneq H$. By maximality of K in Γ we must have $H \cap E \neq 0$. Then $E \cap H/K \neq 0$, so that F/K is an essential extension of E. Given that E has no proper essential extensions, the map $\varphi \colon E \longrightarrow F/K$ must be an isomorphism, so that every element $f \in F$ can be written as $f = e + k$, with $e \in E$ and $k \in K$. Also, $K \cap E = 0$. This means that $F = E \oplus K$, and since by Theorem 4.17 from Homological Algebra a direct summand of an injective module is injective, then E must be injective. □

Theorem 5.17. *Let M be an R-module. Let I be an injective R-module with $M \subseteq I$ and E a maximal essential extension of M contained in I. Then E is an injective module. Moreover, E is unique up to isomorphism and independent of the choice of I.*

Proof. By Theorem 5.16, all we have to show is that E has no proper essential extensions. Let $E \subseteq F$ be an essential extension of E and let i be the inclusion map $E \subseteq I$.

$$\begin{array}{ccccc} 0 & \longrightarrow & E & \xrightarrow{\ j\ } & F \\ & & {\scriptstyle i}\downarrow & \swarrow_{k} & \\ & & I & & \end{array}$$

As I is an injective module and j is injective, we can find k such that the diagram above commutes. Consider $K := \ker(k)$. Since $kj = i$ is injective, we conclude that $K \cap E = K \cap \operatorname{im}(j) = 0$. On the other hand, $E \subseteq F$ is an essential extension, so $K \cap E = 0$ implies $K = 0$ and thus k is injective. Therefore,

$$E \cong j(E) \subseteq F \cong k(F) \subseteq I.$$

Thus $M \subseteq E \subseteq F \subseteq I$, where $M \subseteq E$ and $E \subseteq F$ are both essential extensions. By Theorem 5.8, F is also an essential extension of M. Since we assumed E to be a maximal essnetial extension of M inside I, we conclude that $E = F$, and E has no proper essential extensions. By Theorem 5.16, E is injective.

Now we will show that E is unique up to isomorphism and independent of the choice of I. Let I and I' be two injective modules containing M, and let $E \subseteq I$ and $E' \subseteq I'$ be maximal essential extensions of M inside I and I', respectively.

Since E' is an injective module, there exists $\varphi : E \longrightarrow E'$ making the following diagram commute:

$$\begin{array}{ccccc} 0 & \longrightarrow & M & \longrightarrow & E \\ & & {\scriptstyle i'}\downarrow & \swarrow_{\varphi} & \\ & & E' & & \end{array}$$

Since $i' = \varphi \circ i$ is injective,

$$\ker(\varphi) \cap M = \ker(\varphi) \cap i(M) = 0.$$

Since E is an essential extension of M and $\ker(\varphi)$ is a submodule of E, we conclude that $\ker(\varphi) = 0$ and φ is injective.

Moreover, since E is injective, by Theorem 4.34 from Homological Algebra the short exact sequence

$$0 \longrightarrow E \xrightarrow{\varphi} E' \longrightarrow E'/\varphi(E) \longrightarrow 0.$$

splits, so $E' = \varphi(E) \oplus H$ for some submodule H of E' with $H \cap \varphi(E) = 0$. Moreover, identifying M with its image $i'(M) \subseteq E'$, we have

$$M = i'(M) \subseteq \varphi(E).$$

Therefore, $H \cap M = 0$. Since $M \subseteq E'$ is an essential extension and H, we conclude that $H = 0$, and thus $E = \varphi(E)$ and φ is an isomorphism. □

Theorem 5.18. *Let $M \subseteq E$ be R-modules. The following are equivalent:*

1) *E is a maximal essential extension of M.*

2) *E is injective and an essential extension of M.*

3) *E is a minimal injective module containing M.*

Proof. 1 ⇒ 2: Suppose E is a maximal essential extension of M. Then E has no proper essential extensions by Theorem 5.8 and thus E is injective by Theorem 5.16.

2 ⇒ 3: Given any essential extension $M \subseteq E$ with E injective, if I is an injective module with $M \subseteq I \subseteq E$, then by Theorem 5.8 $I \subseteq E$ is also an essential extension, but I has no proper essential extensions by Theorem 5.16, so $I = E$. Therefore, E is a minimal injective module containing M.

3 ⇒ 1: Suppose E is a minimal injective module containing M, and let I be a maximal essential extension of M inside E. By Theorem 5.17, I is injective, which by minimality of E leads us to $I = E$. In particular, $M \subseteq E$ is an essential extension. Moreover, if there exists an essential extension $M \subseteq I$ such that $M \subseteq E \subseteq I$, then by Theorem 5.8 we conclude that $E \subseteq I$ is also an essential extension. However, E is injective, and thus has no proper essential extensions by Theorem 5.16, so $E = I$. We conclude that E is a maximal extension of M. □

Definition 5.19 (Injective Envelope)**.** Let M be an R-module. The **injective envelope** of M, denoted by $E(M)$, is the isomorphism class of modules satisfying the equivalent conditions of Theorem 5.18.

Remark 5.20. Let M be any R-module. By Theorem 4.33 from Homological Algebra, there exists an injective R-module I containing M. Let E be a maximal essential extension of M contained in I, which exists by Theorem 5.14. Then E is an injective envelope of M. Notice that the injective envelope is defined only up to isomorphism, but Theorem 5.17 guarantees that it is well-defined, not depending on the choice of I or E. Thus, although this is an abuse of notation, we may write $E(M) = E$. This shows that every module has an injective envelope. More generally, Theorem 5.17 says that any injective module containing M must also contain a copy of $E(M)$.

Corollary 5.21. *Let R be a ring and M an R-module. If $M \subseteq I$ is an essential extension where I is an injective R-module, then $E(M) = I$. In particular, $E(I) = I$ for every injective R-module I.*

Proof. By Theorem 5.17, $I = E(M)$ since I is a maximal essential extension of M inside I. □

Lemma 5.22. *For all essential extensions $M \subseteq N$ of R-modules, N embeds into $E(M)$ and $E(M) = E(N)$.*

Proof. By Theorem 5.8 applied to $M \subseteq N \subseteq E(N)$, $M \subseteq E(N)$ is an essential extension. Since $E(N)$ is injective, by Corollary 5.21 we conclude that $E(M) = E(N)$. □

Remark 5.23. If $M \cong N$ then $E(M) = E(N)$ by Lemma 5.22.

Example 5.24. Consider $\mathbb{Z} \subseteq \mathbb{Q}$, which we saw in Example 5.5 is an essential extension. By Theorem 4.20 from Homological Algebra, $\mathbb{Q}$ is an injective $\mathbb{Z}$-module, so we conclude that $E(\mathbb{Z}) = \mathbb{Q}$.

Example 5.25. Consider the ring $\mathbb{Z}$, any prime integer p, and the $\mathbb{Z}$-module

$$\mathbb{Z}\left[\frac{1}{p^\infty}\right] = \left\{\frac{a}{p^n} + \mathbb{Z} \in \mathbb{Q}/\mathbb{Z} \mid a \in \mathbb{Z} \text{ and } n \geqslant 1\right\}.$$

We saw in Example 5.7 that the inclusion

$$\mathbb{Z}/(p^n) \subseteq \mathbb{Z}\left[\frac{1}{p^\infty}\right]$$

defined by $a + (p^n) \mapsto \frac{a}{p^n} + \mathbb{Z}$ is an essential extension. Given any $y \in \mathbb{Z}\left[\frac{1}{p^\infty}\right]$ and any nonzero $n \in \mathbb{Z}$, fix $m \geqslant 1$ and $1 \leqslant a \leqslant p^m - 1$ such that $y = \frac{a}{p^m} + \mathbb{Z}$. Then $a \in \mathbb{Z}$ satisfies

$$a\left(\frac{1}{p^m} + \mathbb{Z}\right) = \frac{a}{p^m} + \mathbb{Z} = y.$$

Therefore, $\mathbb{Z}\left[\frac{1}{p^\infty}\right]$ is a divisible $\mathbb{Z}$-module, and thus injective by Theorem 4.28 from Homological Algebra. By Corollary 5.21, we conclude that

$$E(R/(p^n)) = \mathbb{Z}\left[\frac{1}{p^\infty}\right].$$

Lemma 5.26. *Let R be a noetherian ring, M an R-module and S a multiplicative set in R. Then $E(M_S) = E(M)_S$.*

Proof. By construction, $M \subseteq E(M)$ is an essential extension, and thus $M_S \subseteq E(M)_S$ is an essential extension by Theorem 5.13. By Theorem 5.15, $E(M)_S$ is an injective R_S-module, and by Corollary 5.21, any essential extension of M_S by an injective module must be the injective envelope of M_S. □

Lemma 5.27. *Let A and B be R-modules. Then*

$$E(A \oplus B) = E(A) \oplus E(B).$$

Proof. Since $A \subseteq E(A)$ and $B \subseteq E(B)$, then $A \oplus B \subseteq E(A) \oplus E(B)$, and this is an essential extension. Indeed, if $M \subseteq E(A) \oplus E(B)$ is a non-zero submodule, then there exists $a \in E(A)$ and $b \in E(B)$ such that $(a, b) \in M$ and $a \neq 0$ or $b \neq 0$. Let us assume without loss of generality that $a \neq 0$. Then since $A \subseteq E(A)$ is an essential extension, $Ra \cap A \neq 0$, and therefore $ra \in A$. If $b = 0$ then $r(a, b) = (ra, 0) \in M \cap (A \oplus B)$ and we are done. If $b \neq 0$, then by the same argument we can find $s \in R$ such that $sb \in B$. Clearly, $rsa \in A$ and $rsb \in B$. Then $rs(a, b) \in M \cap (A \oplus B)$. Therefore, $A \oplus B \subseteq E(A) \oplus E(B)$ is an essential extension.

But $E(A) \oplus E(B)$ is a direct sum of two injective modules, and by Theorem 4.16 from Homological Algebra $E(A) \oplus E(B)$ is injective. By Corollary 5.21,

$$E(A \oplus B) = E(A) \oplus E(B).$$ □

Remark 5.28. By Theorem 4.19 from Homological Algebra, arbitrary direct sums of injective modules over noetherian ring are injective. Thus we can extend the Lemma 5.27 to arbitrary direct sums, as long as R is noetherian.

We can now compute the injective envelope of any finitely generated abelian group.

Example 5.29. Any finitely generated abelian group is of the form

$$M \cong \mathbb{Z}^n \oplus \mathbb{Z}/(p_1^{a_1}) \oplus \cdots \oplus \mathbb{Z}/(p_m^{a_m})$$

with $n \geqslant 0$, $p_1, \ldots, p_m$ primes, and $a_1, \ldots, a_m \geqslant 1$. We saw in Example 5.24 and Example 5.25 that

$$E(\mathbb{Z}) = \mathbb{Q} \text{ and } E(\mathbb{Z}/(p^n)) = \mathbb{Z}\left[\frac{1}{p^\infty}\right].$$

Using Lemma 5.27, we conclude that

$$\begin{aligned} E(M) &= E(\mathbb{Z})^n \oplus (\mathbb{Z}/(p_1^{a_1})) \oplus \cdots \oplus (\mathbb{Z}/(p_m^{a_m})) \\ &= \mathbb{Q}^n \oplus \mathbb{Z}\left[\frac{1}{p_1^\infty}\right] \oplus \cdots \oplus \mathbb{Z}\left[\frac{1}{p_m^\infty}\right]. \end{aligned}$$

Theorem 5.30. *Let R be a noetherian ring and $M \neq 0$ a finitely generated R-module. Then*

$$\mathrm{Ass}(M) = \mathrm{Ass}(E(M)).$$

Proof. This is a special case of Theorem 5.9, since $M \subseteq E(M)$ is an essential extension. □

We will soon be looking more closely at the injective hulls of the cyclic modules R/P, where P is a prime ideal in R. For that, we will need a few useful facts.

Corollary 5.31. *Let R be a noetherian ring. For all prime ideals P, $\mathrm{Ass}(E(R/P)) = \{P\}$.*

Proof. We have

$$\begin{aligned} \mathrm{Ass}(E(R/P)) &= \mathrm{Ass}(R/P) && \text{by Theorem 5.30} \\ &= \{P\} && \text{by Lemma 6.21 from Commutative Algebra I.} \end{aligned}$$

□

Corollary 5.32. *Let R be a noetherian ring. For any two distinct prime ideals P and Q, $E(R/P) \ncong E(R/Q)$.*

Proof. By Corollary 5.31,

$$\mathrm{Ass}(E(R/P)) = \{P\} \neq \{Q\} = \mathrm{Ass}(E(R/Q)).$$

Therefore, $E(R/P) \ncong E(R/Q)$. □

Remark 5.33. Fix a prime ideal P and a nonzero submodule $M \subseteq E(R/P)$. By Lemma 6.28 from Commutative Algebra I and Corollary 5.31,

$$\mathrm{Ass}(M) \subseteq \mathrm{Ass}(E(R/P)) = \{R/P\}.$$

Since $\mathrm{Ass}(M) \neq \emptyset$ by Theorem 6.26 from Commutative Algebra I, we conclude that

$$\mathrm{Min}(M) = \mathrm{Ass}(M) = \{R/P\}.$$

Lemma 5.34. *Let R be a noetherian ring and M an R-module such that $\mathrm{Ass}(M) = \{P\}$. Then every element in M is P-torsion, that is, killed by a power of P.*

Proof. Fix a nonzero element $x \in M$. Since

$$\emptyset \neq \mathrm{Min}(R\,x) \subseteq \mathrm{Ass}(R\,x) \subseteq \mathrm{Ass}(M) = \{P\},$$

we conclude that

$$\mathrm{Min}(Rx) = \{P\}.$$

In particular, P is the unique minimal prime over $\mathrm{ann}(x)$, and thus $\sqrt{\mathrm{ann}(x)} = P$. Since R is noetherian, there exists some N such that $P^N \subseteq \mathrm{ann}(x)$, and therefore x is killed by P^N. □

Corollary 5.35. *Let R be a noetherian ring and P a prime ideal in R. Every element of $E(R/P)$ is killed by a power of P.*

Proof. By Remark 5.33, $\mathrm{Ass}(E(R/P)) = \{P\}$. Apply Lemma 5.34. □

Notation 3. Given a prime ideal P in R, we write

$$\kappa(P) := R_P/P_P$$

for the residue field of R_P.

Lemma 5.36. *Let R be a noetherian ring. For any prime ideal P,*

$$E_R(R/P) \cong E_{R_P}(\kappa(P)) \quad \textit{and} \quad E_{R/P}(R/P) \cong E_{\kappa(P)}(\kappa(P)) \cong \kappa(P).$$

Proof. Let $E = E_R(R/p)$. We will first show that the map

$$\begin{aligned} E &\xrightarrow{\varphi} E_P \\ e &\longmapsto \tfrac{e}{1} \end{aligned}$$

is an isomorphism. To do that, let $e \in E$ and suppose $\frac{e}{1} = \phi(e) = 0$. Then there exists $s \notin P$ such that $se = 0$. By Corollary 5.31, $\operatorname{Ass}_R(E) = \{P\}$, and thus every zerodivisor on E is contained in P by Theorem 6.27 from Commutative Algebra I. Hence, s is a regular element on E and thus $e = 0$. This shows ϕ is injective.

Now let $\frac{e}{s} \in E_p$, where $s \notin P$. By Remark 5.33,

$$\operatorname{Ass}_R(Re) = \{P\}$$

and so s is a regular element on Re by Theorem 6.27 from Commutative Algebra I. Thus

$$\begin{aligned} Re &\xrightarrow{f} Re \\ re &\longmapsto sre \end{aligned}$$

is injective. Now let $i : Re \to E$ be the inclusion map. Since E is injective, there exists a map $h : Re \to E$ such that $i = hf$, meaning that it makes the following diagram commute:

$$\begin{array}{ccccc} 0 & \longrightarrow & Re & \xrightarrow{f} & Re \\ & & {\scriptstyle i'}\downarrow & \swarrow_{h} & \\ & & E & & \end{array}$$

Let $e' = h(e)$. Then

$$e = i(e) = hf(e) = h(se) = sh(e) = se'.$$

Therefore,

$$\frac{e}{s} = \frac{se'}{s} = \frac{e'}{1} = \phi(e').$$

Hence, ϕ is surjective and $E_R(R/P) \cong E_R(R/P)_P$. By Theorem 5.15,

$$E_R(R/P)_P \cong E_{R_P}((R/P)_P) \cong E_{R_P}(\kappa(p)).$$

Applying the first isomorphism to $E_{R/P}(R/P)$ shows that

$$E_R(R/p) \cong E_{R/P}(R/P) \cong \left(E_{R/P}(R/P)\right)_P.$$

By Lemma 5.26,

$$\left(E_{R/P}(R/P)\right)_P \cong E_{(R/P)_P}((R/P)_P) = E_{\kappa(P)}(\kappa(P)).$$

By Corollary 5.11, since $\kappa(P)$ is a field then $\kappa(P)$ has no proper essential extensions of $\kappa(P)$-modules, and thus $E(\kappa(P)) = \kappa(P)$. □

Lemma 5.37. *Let R be a noetherian ring and consider finitely generated R-modules $A \subseteq B$. The following are equivalent:*

1) *$A \subseteq B$ is an essential extension.*

2) *$A_P \subseteq B_P$ is an essential extension for all primes P.*

3) *$(0 :_{A_P} PR_P) = (0 :_{B_P} PR_P)$ for all primes P.*

4) *The natural map*
$$\mathrm{Hom}_{R_P}(\kappa(P), A_P) \longrightarrow \mathrm{Hom}_{R_P}(\kappa(P), B_P)$$
induced by applying $\mathrm{Hom}_{R_P}(\kappa(P), -)$ to $A \subseteq B$ is an isomorphism for all primes P.

Proof. 1 $\Rightarrow$ 2 is Theorem 5.13.

2 $\Rightarrow$ 3: It suffices to show that if $(R, \mathfrak{m})$ is a noetherian local ring and $A \subseteq B$ is an essential extension of R-modules, then $(0 :_A \mathfrak{m}) = (0 :_B \mathfrak{m})$. First, note that if $A \subseteq B$ is an essential extension, then given any $x \in (0 :_B \mathfrak{m})$, the fact that $A \subseteq B$ is an essential extension implies that there exists $a = rx \in A \cap (Rx)$. Then
$$a\mathfrak{m} = r(x\mathfrak{m}) = 0 \implies a \in (0 :_A \mathfrak{m}).$$
and thus $Rx \cap (0 :_A \mathfrak{m}) \neq 0$. We conclude that $(0 :_A \mathfrak{m}) \subseteq (0 :_B \mathfrak{m})$ is an essential extension of $R/\mathfrak{m}$-modules. On the other hand, $R/\mathfrak{m}$ is a field, and thus by Corollary 5.11 there are no proper essential extensions over k. We conclude that $(0 :_A \mathfrak{m}) = (0 :_B \mathfrak{m})$.

3 $\iff$ 4: For all R-modules N and all prime ideals P, we have a natural isomorphism
$$(0 :_{N_P} PR_P) \cong \mathrm{Hom}_{R_P}(R/PR_P, N_P) = \mathrm{Hom}_{R_P}(\kappa(P), N_P).$$
Moreover, $\mathrm{Hom}_{R_P}(\kappa(P), -)$ is left exact, and the map injective
$$\mathrm{Hom}_{R_P}(\kappa(P), A_P) = (0 :_{A_P} PR_P) \xrightarrow{\varphi_P} (0 :_{B_P} PR_P) \cong \mathrm{Hom}_{R_P}(\kappa(P), B_P)$$
induced by applying $\mathrm{Hom}_{R_P}(\kappa(P), -)$ to $A_P \subseteq B_P$ has image the submodule $(0 :_{A_P} PR_P)$ of $(0 :_{B_P} PR_P)$. Thus
$$(0 :_{A_P} PR_P) = (0 :_{B_P} PR_P) \iff \varphi_P \text{ is an iso for all primes.}$$

3 $\Rightarrow$ 1: Let M be a nonzero submodule of B and consider $P \in \mathrm{Ass}(M)$. Fix $a \in M$ such that $P = (0 :_R a)$. Then
$$PR_P = (0 :_R a)_P = \left(0 :_{R_P} \frac{a}{1}\right)$$
so
$$\frac{a}{1} \in (0 :_{B_p} PR_p) = (0 :_{A_P} PR_P).$$
Thus, $\frac{a}{1} = \frac{u}{s}$ for some $u \in A$ and $s \notin P$, so there exists some $t \notin P$ such that $t(u - sa) = 0$. Set $x := tu = tsa$. Since $u \in A$ and $a \in M$ we conclude that $x \in M \cap A$. Moreover, $x \neq 0$ since $t, s \notin P$ and thus $ts \notin P = (0 :_R a)$. $\square$

5.3 The structure of injectives

Definition 5.38 (Indecomposable)**.** An R-module I is said to be **indecomposable** if it cannot be written as the direct sum of two nonzero submodules.

Lemma 5.39. *Any injective submodule of an injective module E is a direct summand of E.*

Proof. Suppose that $I \subseteq E$ are both injective modules. Since I is injective, the short exact sequence

$$0 \longrightarrow I \longrightarrow E \longrightarrow E/I \longrightarrow 0$$

splits, by Theorem 4.34 from Homological Algebra, and thus $E \cong I \oplus E/I$ and I is a direct summand of E. □

Theorem 5.40. *Let R be a noetherian ring. An injective R-module E is indecomposable if and only if E has no proper injective submodules.*

Proof. By Lemma 5.39, any injective submodule of E is a direct summand of E. By Theorem 4.17 from Homological Algebra, any direct summand of E is injective. Thus E has proper direct summands if and only if it contains proper injective submodules. □

Theorem 5.41. *Let R be a noetherian ring. An injective R-module E is indecomposable if and only if $E = E(R/P)$ for some $P \in \operatorname{Spec}(R)$.*

Proof.

($\Rightarrow$) Let $P \in \operatorname{Ass}(E)$, so that $P = \operatorname{ann}(e)$ for some nonzero $e \in E$. Then

$$R/P \cong Re \subseteq E,$$

Since E is injective, then by definition of injective envelope we have

$$Re \subseteq E(Re) = E(Q/P) \subseteq E.$$

By Theorem 5.40, E has no proper injective submodules, so $E = E(R/P)$.

($\Leftarrow$) We will show that $E = E(R/P)$ is indecomposable. If not, then $E_R(R/P) = E_1 \oplus E_2$ for two nonzero submodules E_1 and E_2 of E with $E_1 \cap E_2$. Since $R/P \subseteq E(R/P)$ is an essential extension, then

$$J_1 := E_1 \cap R/P \neq 0 \text{ and } J_2 := E_2 \cap R/P \neq 0.$$

These are both R-submodules of R/P, and thus we can view then as ideals in the domain R/P. But since R/P is a domain,

$$J_1 \cap J_2 \supseteq J_1 J_2 \neq 0.$$

This contradicts the assumption that $E_1 \cap E_2 = 0$, so $E(R/P)$ is indecomposable. □

Theorem 5.42. *Let R be a noetherian ring. Every injective R-module is a direct sum of indecomposable injective R-modules.*

Proof. Let I be an injective R-module. Consider the set T of all indecomposable injective submodules of I, and the set S of all subsets of T with the following property: if $F \in S$, then the sum of all modules in F is direct, meaning that

$$\sum_{M \in F} M = \bigoplus_{M \in F} M.$$

The set S is partially ordered by inclusion, and will show that we can apply Zorn's Lemma to it.

First let us check that S is nonempty. Pick $P \in \mathrm{Ass}(I)$, which must be the annihilator of some non-zero $a \in I$. Note that $R/P \cong R\,a \subseteq E$ and E is injective, so by Theorem 5.17 $E(R/P) \subseteq E$. By Theorem 5.41, $E(R/P)$ is an indecomposable injective submodule of I. In particular, $E(R/P) \in T$, and thus $\{E(R/P)\} \in S$. We conclude that S is nonempty.

If $\{F_\lambda\}_{\lambda \in \Lambda}$ is a family in S totally ordered by inclusion, then we claim that

$$F := \bigcup_{\lambda \in \Lambda} F_\lambda \in S.$$

To prove that, we need to show that the sum of all the modules in F is a direct sum. This is a statement only involves finitely many modules at a time: given

$$a_{\lambda_1} + \ldots + a_{\lambda_n} = 0$$

with each $a_{\lambda_j} \in M_{\lambda_j}$ for distinct modules $M_{\lambda_j} \in F_{\lambda_j}$, we need to prove that $a_{\lambda_j} = 0$ for all j. There exists $\lambda \in \Lambda$ large enough such that $M_{\lambda_1}, \ldots, M_{\lambda_n} \in F_\lambda$, but since $F_\lambda \in S$, we know that the sum of the modules in F_λ is direct, and since

$$a_{\lambda_1} + \ldots + a_{\lambda_n} = 0$$

is an element in that direct sum with each a_{λ_i} coming from a different module in the sum, we conclude that all the a_{λ_j} must be zero. This shows that the sum of the modules in F is direct, and therefore that $F \in S$. Moreover, F is an upper bound of the chain $\{F_\lambda\}_{\lambda \in \Lambda}$. Thus Zorn's Lemma applies to S, and S has a maximal element F.

Let E be the direct sum of all modules in F. Arbitrary direct sums of injective modules over a noetherian ring are injective, by Theorem 4.19 from Homological Algebra, and thus E is injective. Moreover, since $E \subseteq I$, then E is a direct summand of I by Lemma 5.39. Consider H such that $I = E \oplus H$. We will show that $H = 0$, which proves that $I = E$.

If $H \neq 0$, then consider $Q \in \mathrm{Ass}(H)$, which exists by Theorem 6.26 from Commutative Algebra I. Fix $x \in H$ such that $Q = \mathrm{ann}(Rx)$. Since $R\,x \cong R/Q$ and H is injective, then H contains a copy of the injective envelope of Rx, so

$$E(R/Q) = E(R\,x) \subseteq H \subseteq E.$$

Since $E(R/Q)$ is injective and R is noetherian, then $E(R/Q)$ is a direct summand of H and of E, by Lemma 5.39; say that $H = E(R/Q) \oplus G$. Then

$$I = E \oplus H = (E \oplus E(R/Q)) \oplus G.$$

Therefore, we may enlarge F by adding $E(R\,x) = E(R/Q)$, which is impossible by the maximality of F. Then $H = 0$ and $I = E$, which is a direct sum of indecomposable injective modules. □

Corollary 5.43. *Let R be a noetherian ring. Any injective R-module E is of the form*

$$E = \bigoplus_{P \in \operatorname{Ass}(E)} (E(R/P))^{\mu_P}$$

where μ_p are cardinals.

Proof. By Theorem 5.42, E is an arbitrary direct sum of indecomposable injective modules. By Theorem 5.41, the indecomposable injective modules are precisely the modules $E(R/P)$ for P some prime ideal in R. Moreover, if E has a submodule isomorphic to $E(Q/P)$, then by Corollary 5.31 we have

$$P \in \operatorname{Ass}(E(R/P)) \subseteq \operatorname{Ass}(E).$$

Conversely, if $P \in \operatorname{Ass}(E)$, then there is an inclusion $R/P \subseteq E$, so E contains a copy of its injective hull, meaning $E(R/P) \subseteq E$. Since $E(R/P)$ and E are both injective, we conclude that $E(R/P)$ must be a direct summand of E by Lemma 5.39. □

Lemma 5.44. *Let M be a nonzero R-module with $I \subseteq \operatorname{ann}(M)$. Then*

$$\operatorname{Hom}_R(R/I, E_R(M)) \cong E_{R/I}(M).$$

Proof. Set $E = E_R(M)$, and identify $\operatorname{Hom}_R(R/I, E_R(M))$ with the submodule of E given by

$$E_I = (0 :_E I) = \{e \in E \mid Ie = 0\}.$$

First, note that $M \subseteq E$ and $I \subseteq \operatorname{ann}(M)$ gives us $M \subseteq E_I$. In particular, $E_I \neq 0$.

Now we claim that $M \subseteq E_I$ is an essential extension as both R-modules and R/I-modules. It is an essential extension of R-modules since $M \subseteq E_I \subseteq E$ and $M \subseteq E$ is an essential extension of R-modules, by Theorem 5.8. Moreover, the R/I-submodules of E_I are the R-submodules killed by I, and thus this is also an essential extension of R/I-modules.

Finally, we claim that E_I is an injective module over R/I. Consider the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B \\ & & \downarrow & & \\ & & E_I & & \end{array}$$

where A and B are R/I-modules and f is injective.

Since E is an injective R-module, there exists a map φ making the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B \\ & & \downarrow & & \vdots \varphi \\ & & E_I & \hookrightarrow & E \end{array}$$

commute. Moreover, since B is also an R/I-module, we must have $IB = 0$, and therefore $I(\mathrm{im}(\varphi)) = 0$. Then $\mathrm{im}(\varphi) \subseteq E_I$, meaning that we can write φ as a map $\varphi \colon B \longrightarrow E_I$. This shows that E_I is injective, by Baer's Criterion Theorem 4.18 from Homological Algebra.

Therefore, $R/I \subseteq E_I$ is an essential extension, both as R-modules and as R_I-modules, and E_I is an injective R/I-module, so we must have $E_I = E_{R/I}(M)$. □

Lemma 5.45. *Let R be a noetherian ring, P a prime ideal, and I any ideal. Then*

$$\mathrm{Hom}_R(R/I, E_R(R/P)) = \begin{cases} E_{R/I}(R/P), & \text{if } I \subseteq P \\ 0, & \text{if } I \nsubseteq P. \end{cases}$$

Proof. If $I \subseteq P$, then R/P is an R/I-module. By Lemma 5.44,

$$\mathrm{Hom}_R(R/I, E_R(R/P)) \cong E_{R/I}(R/P).$$

If $I \nsubseteq P$, let $s \in I$ with $s \notin P$. By Corollary 5.31, P is the only associated prime of $E_R(R/P)$, and thus $s \notin P$ implies that s is regular on $E_R(R/P)$. Fix a homomorphism of R-modules $f \colon R/I \longrightarrow E_R(R/P)$. For all $r \in R$,

$$sf(r + I) = f(sr + I) = f(0) = 0.$$

Since s is regular on $E_R(R/P)$, we conclude that $f = 0$ and $\mathrm{Hom}_R(R/I, E_R(R/P)) = 0$. □

In what follows, we will need the following technical but useful lemma:

Lemma 5.46. *Let M be a finitely generated R-module and $\{N_i\}_{i \in I}$ be an arbitrary collection of R-modules. Then*

$$\mathrm{Hom}_R\left(M, \bigoplus N_i\right) \cong \bigoplus_i \mathrm{Hom}_R(M, N_i).$$

Proof. Consider the map of R-modules $F : \bigoplus_i \mathrm{Hom}_R(M, N_i) \longrightarrow \mathrm{Hom}_R\left(M, \bigoplus N_i\right)$ that sends each $f = \{f_i\}_i$ with $f_i \in \mathrm{Hom}_R\left(M, N_i\right)$ to $F(f) \in \mathrm{Hom}_R(M, \bigoplus_i N_i)$ with

$$F(f)(m) = f_i(m).$$

Injectivity is immediate from the definition: if $F(f) = 0$, then we must have $f_i = 0$ for all i, and thus $f = 0$.

Now fix generators $m_1, \dots, m_n$ for M. Given $g \in \mathrm{Hom}_R(M, \bigoplus N_i)$, note that

$$g(m_j) = \{a_i\}_{i \in I}$$

has $a_i = 0$ for all but finitely many indices. Collecting all such indices for $g(m_1), \dots, g(m_n)$, there is a finite set of indices, say $i_1, \dots, i_d$ such that

$$\mathrm{im}(g) \subseteq N_{i_1} \oplus \cdots \oplus N_{i_d}.$$

For each $i \in \{i_1, \dots, i_d\}$, set $h_i \in \mathrm{Hom}_R(M, N_i)$ to be the ith coordinate of g:

$$h_i(m) = g(m)_i.$$

For all other i, set $h_i = 0$. Then $h = \{h_i\}$ satisfies $F(h) = g$, showing that F is surjective. We conclude that F is an isomorphism. □

We now address the question of well-definedness of the decomposition of an injective module into a direct sum of indecomposables $E(R/P)$: it turns out that the number of copies of $E(R/P)$ does not depend on the choice of decomposition.

Theorem 5.47. *Let R be a noetherian ring and I an injective module. For all primes P, the number of copies of $E_R(R/P)$ appearing in any decomposition of I into indecomposables is uniquely determined. More precisely, if*

$$I \cong E_R(R/P)^\alpha \oplus M,$$

with $E_R(R/P)$ not a direct summand of M, then

$$\alpha = \mathrm{rank}_{\kappa(P)} \left(\mathrm{Hom}_{R_P}(\kappa(P), I_P) \right).$$

Proof. By Corollary 5.43, M decomposes as a direct sum of injective hulls of R/Q, where each Q is a prime ideal. Let $E(R/Q)$ be a direct summand of M. By assumption, $P \neq Q$. If $P \not\subseteq Q$ then by Lemma 5.45

$$\mathrm{Hom}_R(R/P, E_R(R/Q)) = 0.$$

If $P \subsetneq Q$, then $(R/Q)_P = 0$. By Theorem 5.17 and Theorem 5.13, $(R/Q)_P \subseteq E(R/Q)_P$ is an essential extension, and thus we conclude that $E(R/Q)_P = 0$. By Theorem 3.77 from Homological Algebra we have

$$\mathrm{Hom}_R(R/P, E_R(R/Q))_P \cong \mathrm{Hom}_{R_P}(\kappa(P), E_R(R/Q)_P) = 0.$$

Therefore, for every indecomposable direct summand $E(R/Q)$ of M, we have

$$\mathrm{Hom}_R(R/P, E_R(R/Q))_P = 0.$$

Since R/P is finitely generated, by Lemma 5.46 the functor $\mathrm{Hom}_R(R/P, -)$ commutes with taking arbitrary direct sums, and thus $\mathrm{Hom}_{R_P}(\kappa(P), M_P)$ is a direct sum of terms of the form

$$\mathrm{Hom}_{R_P}(\kappa(P), (E(R/Q))_P) = 0.$$

We conclude that

$$\mathrm{Hom}_{R_P}(\kappa(P), M_P) = 0$$

and thus

$$\mathrm{Hom}_{R_P}(\kappa(P), I_P) \cong \mathrm{Hom}_{R_P}(\kappa(P), E_R(R/P)_P^\alpha).$$

Therefore,

$$\begin{aligned}
\mathrm{Hom}_R(R/P, I)_P &\cong \mathrm{Hom}_{R_P}(\kappa(P), I_P) \\
&\cong \mathrm{Hom}_R(R/P, E_R(R/P))_P^\alpha \\
&\cong E_{R/P}(R/P)_P^\alpha && \text{by Lemma 5.45} \\
&\cong E_{\kappa(P)}(\kappa(P))^\alpha && \text{by Lemma 5.36} \\
&\cong \kappa(P)^\alpha && \text{by Lemma 5.36.}
\end{aligned}$$

This shows that $\mathrm{Hom}_R(R/P, I)_P$ is a vector space over $\kappa(P)$ of rank α. In particular, the number of copies of $E(R/P)$ that appear in a direct sum decomposition of I is independent of the choice of decomposition. □

Remark 5.48. As we saw in Corollary 5.43, the primes P such that $E(R/P)$ appears in the direct sum decomposition of I are precisely the associated primes of I. We can also rediscover this fact by noting that P is associated to I if and only if P_P is associated to I_P, and so by definition of associated primes we have

$$P \in \mathrm{Ass}(I) \iff P_P \in \mathrm{Ass}(I_P) \iff \mathrm{Hom}_{R_P}(R_P/P_P, I_P) \neq 0.$$

5.4 Minimal Injective resolutions

Definition 5.49 (Minimal injective resolution)**.** Let R be a ring, M an R-module, and

$$E = 0 \longrightarrow E^0 \xrightarrow{\partial^0} E^1 \xrightarrow{\partial^1} \cdots$$

an injective resolution of M. We say that E is a **minimal injective resolution** of M if

$$E^0 \cong E(M) \quad \text{and} \quad E^i \cong E(\operatorname{im}(\partial^{i-1})) \text{ for all } i \geqslant 1.$$

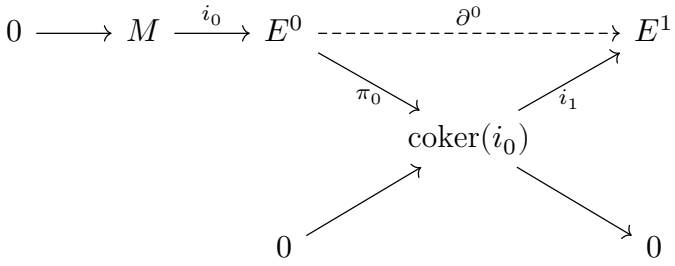
The existence of injective envelopes together with the construction of injective resolutions we saw in Homological Algebra imply that every module has a minimal injective resolution:

Theorem 5.50. *Every R-module has a minimal injective resolution.*

Proof. By Theorem 5.17, every R-module has an injective envelope. We start by embedding M into its injective envelope $E^0 = E(M)$, and consider the cokernel of the inclusion:

$$0 \longrightarrow M \xrightarrow{i_0} E^0 \xrightarrow{\pi_0} \operatorname{coker}(\partial^0) \longrightarrow 0.$$

Now embed $\operatorname{coker}(\partial^0)$ in its injective hull $E^1 = E(\operatorname{coker}(\partial^0))$:



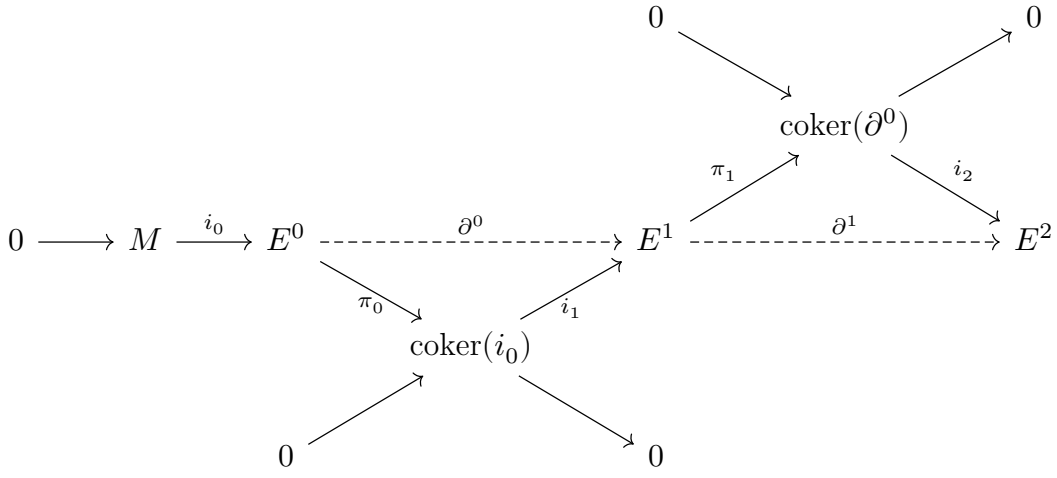
Take $\partial^0 := i_1\pi_0$. Since i_1 is injective,

$$\ker(\partial^0) = \ker(i_1\pi_0) = \ker(\pi_0) = \operatorname{im}(i_0).$$

Notice also that

$$\operatorname{coker}(i_0) = \operatorname{im}(\partial^0) = \ker(E^1 \longrightarrow \operatorname{coker}\partial^0).$$

So we can now we continue in a similar fashion, by taking the injective hull E^2 of $\operatorname{coker}(\partial^0)$, and so on.



By construction and since i_2 is injective, $\ker(\partial^1) = \operatorname{im}(\partial^0)$, and our complex is exact at E^1. The process continues analogously.

At each stage, we take

$$E^{i+1} = E(\operatorname{coker}(\partial^{i-1})) \quad \text{and} \quad \operatorname{im}(\partial^i) = \operatorname{coker}(\partial^{i-1})$$

so

$$E^i = E(\operatorname{im}(\partial^{i-1})).$$ □

Theorem 5.51. *Let R be a noetherian ring, M an R-module, and I a minimal injective resolution of M. For any multiplicatively closed subset S of R, I_S is a minimal injective resolution of M_S.*

Proof. Let

$$E = 0 \longrightarrow M \longrightarrow E^0 \xrightarrow{\partial^0} E^1 \xrightarrow{\partial^1} \cdots$$

be any minimal free resolution for M. Since localization is exact,

$$E = 0 \longrightarrow M_S \longrightarrow E^0_S \xrightarrow{\partial^0_S} E^1_S \xrightarrow{\partial^1_S} \cdots$$

is also an exact sequence. By Lemma 5.26,

$$E^0_S = E(M)_S \cong E(M_S) = E^0(M_S).$$

For all i, the exactness of localization gives us $\operatorname{im}(\partial^i)_S = \operatorname{im}(\partial^i_S)$, and thus by Lemma 5.26

$$E^{i+1}_S = (E^{i+1}(M))_S = (E(\operatorname{im}(\partial^i))_S = E(\operatorname{im}(\partial^i_S)).$$

Therefore, E_S is a minimal injective resolution for M_S. □

Definition 5.52. Let R be a noetherian ring and M be any R-module. The **ith Bass number** of M with respect to P is

$$\mu_i(P, M) := \operatorname{rank}_{\kappa(P)} \left(\operatorname{Ext}^i_{R_P}(\kappa(P), M_P) \right).$$

Remark 5.53. By definition, $\mu_i(P, M) \neq 0$ if and only if

$$\operatorname{Ext}^i_{R_P}(\kappa(P), M_P) \neq 0.$$

By Theorem 6.35 from Homological Algebra, since R/P is finitely generated we have

$$\operatorname{Ext}^i_R(R/P, M)_P \cong \operatorname{Ext}^i_{R_P}(\kappa(P), M_P).$$

Therefore, $\operatorname{Ext}^i_R(R/P, M)_P \neq 0$, and thus $\operatorname{Ext}^i_R(R/P, M) \neq 0$.

Bass numbers are named after Hyman Bass, honoring his seminal paper *On the ubiquity of Gorenstein rings* [Bas63]. We will now show that the Bass numbers of M completely determine the injective modules that appear in a minimal injective resolution for M.

Theorem 5.54. *Let R be a noetherian ring, M an R-module, and let*

$$E = 0 \longrightarrow E^0 \xrightarrow{\partial^0} E^1 \xrightarrow{\partial^1} \cdots$$

be any minimal free resolution for M. For all $i \geqslant 0$,

$$E^i = \bigoplus_{P \in \operatorname{Spec}(R)} E(R/P)^{\mu_i(P,M)}.$$

Therefore, the modules E^i do not depend on the choice of minimal injective resolution E, and if M is a finitely generated module, then $\mu_i(P, M) < \infty$ for all primes P and all i.

Proof. Consider a minimal injective resolution of M

$$E \quad = 0 \longrightarrow E^0 \xrightarrow{\partial^0} E^1 \xrightarrow{\partial^1} \cdots$$

By Theorem 5.51,

$$E_P = \quad 0 \longrightarrow E^0_P \xrightarrow{\alpha_1} E^1_P \xrightarrow{\alpha_2} E^2_P \xrightarrow{\alpha_3} \cdots$$

is an injective resolution of M_P. Applying $\operatorname{Hom}_{R_P}(\kappa(P), -)$ to E_P, the ith cohomology module is, by definition,

$$\operatorname{Ext}^i_{R_P}(\kappa(P), M_P) \cong \operatorname{H}^i\left(\operatorname{Hom}_{R_P}(\kappa(P), E_P)\right).$$

We claim that $\operatorname{Hom}_{R_P}(\kappa(P), \partial^i_P) = 0$ for all i. Once we prove the claim, it follows that

$$\operatorname{Ext}^i_{R_P}(\kappa(P), M_P) \cong \operatorname{Hom}_{R_P}(\kappa(P), E^i_P).$$

Thus

$$\mu_i(P, M) = \dim_{\kappa(P)}\left(\operatorname{Ext}^i_{R_P}(\kappa(P), M_P)\right) = \dim_{\kappa(P)}\left(\operatorname{Hom}_{R_P}(\kappa(P), E^i(M)_P)\right)$$

and by Theorem 5.47 this dimension counts precisely the number of copies of $E(R/P)$ in a direct sum decomposition of $E^i(M)$.

Fix i. It remains to show that $\partial^*_P = \operatorname{Hom}_{R_P}(\kappa(P), \partial^i_P) = 0$ for all P. Set $C^0 := M$ and $C^i := \operatorname{im}(\partial^{i-1})$ for all $i \geqslant 1$, so that we have an exact sequence

$$0 \longrightarrow C^i \longrightarrow E^i \longrightarrow E^{i+1}.$$

Since localization is exact and $\operatorname{Hom}_{R_P}(\kappa(P), -)$ is left exact, we get an exact sequence

$$0 \longrightarrow \operatorname{Hom}_{R_P}(\kappa(P), C^i_P) \longrightarrow \operatorname{Hom}_{R_P}(\kappa(P), E^i_P) \xrightarrow{\partial^*_P} \operatorname{Hom}_{R_P}(\kappa(P), E^{i+1}_P).$$

Since $E^i = E(C^i)$, the inclusion $C^i \subseteq E^i$ is an essential extension. By Lemma 5.37, the natural map

$$\operatorname{Hom}_{R_P}(\kappa(P), C^i_P) \longrightarrow \operatorname{Hom}_{R_P}(\kappa(P), E^i_P)$$

is an isomorphism for all primes P. From the exact sequence

$$0 \longrightarrow \operatorname{Hom}_{R_P}(\kappa(P), C^i_P) \longrightarrow \operatorname{Hom}_{R_P}(\kappa(P), E^i_P) \xrightarrow{\partial^*_P} \operatorname{Hom}_{R_P}(\kappa(P), E^{i+1}_P).$$

we conclude that $\partial^*_P = \operatorname{Hom}_{R_P}(\kappa(P), \partial^i_P)$ vanishes for all P. □

Lemma 5.55. *Let M be a finitely generated R-module over a noetherian ring R. Then $\mu_i(P, M)$ is finite for all i and all prime ideals P.*

Proof. Since M is a finitely generated R-module, M_P is a finitely generated R-module. Since $\kappa(P) = R_P/P_P$ is also a finitely generated module over R_P we conclude that all the Ext-modules $\operatorname{Ext}^i_{R_P}(\kappa(P), M_P)$ are finitely generated R_P-modules. These Ext-modules are killed by P_P, and thus are also finitely generated over the field $\kappa(P) = R_P/P_P$. We conclude that all the Bass numbers of M are finite. □

Remark 5.56. Given a noetherian ring R and an R-module M, the fact (Theorem 5.51) that any minimal injective resolution E for M localizes to a minimal injective resolution E_P of M_P, together with the fact (Theorem 5.54) that the Bass numbers of M measure the number of copies of $E(R/P)$ in E^i, implies that

$$\mu_i(P, M) = \mu_i(P_P, M_P).$$

More generally, if $P \subseteq Q$ are prime ideals, then

$$\mu_i(PR_Q, M_Q) = \mu_i(P, M).$$

Definition 5.57. Let

$$E \quad = 0 \longrightarrow E^0 \xrightarrow{\partial^0} E^1 \xrightarrow{\partial^1} \cdots$$

be an injective resolution for M. We say that E has **length** n if $E^n \neq 0$ and $E^i = 0$ for all $i \geqslant n+1$.

Definition 5.58. Let R be a noetherian ring and M be an R-module. The **injective dimension** of M is

$$\operatorname{injdim}(M) := \inf\{n \mid M \text{ has a minimal injective resolution of length } n\}.$$

Theorem 5.59. *Let R be a noetherian ring and let M be an R-module.*

$$\operatorname{injdim}(M) = \sup\{i \mid \mu_i(P, M) \neq 0 \text{ for some prime ideal } P\}.$$

Moreover, all minimal injective resolutions of M have length $\operatorname{injdim}(M)$.

Proof. Let

$$E \quad = 0 \longrightarrow E^0 \xrightarrow{\partial^0} E^1 \xrightarrow{\partial^1} \cdots$$

be any minimal injective resolution for M. By Theorem 5.54,

$$E^i = \bigoplus_{P \in \operatorname{Spec}(R)} E(R/P)^{\mu_i(P,M)}.$$

Therefore, $E^i = 0$ if and only if $\mu_i(P, M) = 0$ for all primes P. Since $\mu_i(P, M)$ does not depend on the choice of minimal resolution E, all minimal injective resolutions for M have the same length, which is given by

$$\sup\{i \mid \mu_i(P, M) \neq 0 \text{ for some prime ideal } P\}. \qquad \square$$

As we noted in Remark 5.53, if $\mu_i(P, M) \neq 0$ for some i and some P, then

$$\operatorname{Ext}^i_R(R/P, M) \neq 0.$$

Given any injective resolution I for M, note that

$$\operatorname{Hom}_R(R/P, I_i) = \operatorname{Ext}^i_R(R/P, M) \neq 0 \implies I_i \neq 0.$$

Therefore, the length of I is at least

$$\sup\{i \mid \mu_i(P, M) \neq 0 \text{ for some prime ideal } P\}.$$

The following lemma will soon be useful.

Lemma 5.60. *Let $(R, \mathfrak{m}, k)$ be a noetherian local ring and M an R-module. Fix i. If*

$$\operatorname{Ext}^i_R(k, M) = 0$$

then $\operatorname{Ext}^i_R(L, M) = 0$ for all finite length modules L.

Proof. If $\ell(L) = 1$, then $L \cong k$ and the result holds trivially. Suppose $\ell(L) \geqslant 2$ and that the lemma holds for all modules of length less than $\ell(L)$. Then we can find a submodule of L isomorphic to k, giving us a short exact sequence

$$0 \longrightarrow k \longrightarrow L \longrightarrow C \longrightarrow 0$$

where $\ell(C) = \ell(L) - 1$. By induction hypothesis, $\operatorname{Ext}^i_R(k, C) = 0$. From the long exact sequence

$$\cdots \longrightarrow \underbrace{\operatorname{Ext}^i_R(C, M)}_{0} \longrightarrow \operatorname{Ext}^i_R(L, M) \longrightarrow \underbrace{\operatorname{Ext}^i_R(k, M)}_{0} \longrightarrow \cdots$$

we conclude that $\operatorname{Ext}^i_R(L, M) = 0$. $\square$

Theorem 5.61. *Let R be a noetherian ring and M a finitely generated R-module. Let $P \subseteq Q$ be prime ideals and set $h = \operatorname{height}(Q/P)$. If $\mu_i(P, M) \neq 0$ for some i, then $\mu_{i+h}(Q, M) \neq 0$.*

Proof. By induction on h. When $h = 0$, we must have $P = Q$ and there is nothing to prove. When $h = 1$, we noted in Remark 5.56 that

$$\mu_i(P, M) = \mu_i(PR_Q, M_Q).$$

This allows us to reduce to the case when $(R, \mathfrak{m})$ is a noetherian local ring, $Q = \mathfrak{m}$, and $\operatorname{height}(\mathfrak{m}/P) = 1$. Choose $x \in \mathfrak{m} \setminus P$. Since $x \notin P$ and $\operatorname{height}(\mathfrak{m}/P) = 1$, the maximal ideal $\mathfrak{m}$ is minimal over $P + (x)$, and thus $\dim(R/(P + (x)) = 0$. By Theorem 9.15 from Commutative Algebra I, $\ell_R(R/(P + (x))) < \infty$, which we will use later.

On the other hand, as we noted in Remark 5.53, if $\mu_i(P, M) \neq 0$ then $\operatorname{Ext}^i_R(R/P, M) \neq 0$. Suppose, by way of contradiction, that

$$\mu_{i+h}(\mathfrak{m}, M) = \mu_{i+1}(\mathfrak{m}, M) = 0.$$

Then $\operatorname{Ext}^{i+1}_R(k, M) = 0$. We saw before that $R/(P+(x))$ is a finite length module, and thus by Lemma 5.60 we conclude that $\operatorname{Ext}^{i+1}_R(R/(P + (x)), M) = 0$.

Since x is regular on R/P, we have a short exact sequence

$$0 \longrightarrow R/P \xrightarrow{x} R/P \longrightarrow R/(P + (x)) \longrightarrow 0.$$

Applying $\operatorname{Hom}_R(-, M)$, we get the exact sequence

$$\cdots \longrightarrow \operatorname{Ext}^i_R(R/P, M) \xrightarrow{x} \operatorname{Ext}^i_R(R/P, M) \longrightarrow \underbrace{\operatorname{Ext}^{i+1}_R(R/(P + (x)), M)}_{0} \longrightarrow \cdots.$$

Therefore, $\operatorname{Ext}^i_R(R/P, M) = x \operatorname{Ext}^i_R(R/P, M)$. Since $\operatorname{Ext}^i_R(R/P, M)$ is a finitely generated module, by NAK we conclude that $\operatorname{Ext}^i_R(R/P, M) = 0$, which is a contradiction. We conclude that $\mu_{i+h}(k, M) \neq 0$.

Now consider any $h \geqslant 2$, and suppose that the statement holds for $h - 1$. If $P \subseteq Q$ are prime ideals such that $\operatorname{height}(P/Q) = h$, then we can find a prime ideal $P \subseteq \mathfrak{a} \subseteq Q$ such that $\operatorname{height}(\mathfrak{a}/P) = 1$ and $\operatorname{height}(Q/\mathfrak{a}) = h - 1$. Suppose that $\mu_i(P, M) \neq 0$. By the case $h = 1$, we get $\mu_{i+1}(\mathfrak{a}, M) \neq 0$. By induction hypothesis applied to $\mathfrak{a} \subseteq Q$, we must have

$$\mu_{i+h}(Q, M) = \mu_{(i+1)+(h-1)}(Q, M) \neq 0. \qquad \square$$

Theorem 5.62. *Let $(R, \mathfrak{m}, k)$ be a noetherian local ring and M a finitely generated R-module. Then*

$$\operatorname{injdim}(M) = \sup\{n \mid \operatorname{Ext}^n_R(k, M) \neq 0\}.$$

Proof. Set

$$r = \operatorname{injdim}(M) \qquad \text{and} \qquad \ell = \sup\{n \mid \operatorname{Ext}^n_R(k, M) \neq 0\}.$$

Since $\operatorname{Ext}^i_R(k, M)$ can be computed via a minimal injective resolution for M, we see that $\operatorname{Ext}^i_R(k, M) = 0$ for all $i > r$, and thus $\ell \leqslant r$.

Suppose $\ell < r$. By Theorem 5.59, $\mu_i(P, M) \neq 0$ for some $i > \ell$ and some prime ideal P. By definition of ℓ, for all $n > \ell$ we have

$$\operatorname{Ext}^n_R(k, M) = 0 \implies \mu_n(\mathfrak{m}, M) = 0.$$

Therefore, $P \neq \mathfrak{m}$, and thus $h = \operatorname{height}(\mathfrak{m}/P) = \dim(R/P) \geqslant 1$. By Theorem 5.61, $\mu_{i+h}(\mathfrak{m}, M) \neq 0$. By definition, this says that $\operatorname{Ext}^{i+h}_R(k, M) \neq 0$. This is a contradiction, since $i + h > \ell$ and given the definition of ℓ. $\square$

Corollary 5.63. *Let R be a noetherian ring and M a nonzero finitely generated R-module. Then* $\operatorname{injdim}(M) \geqslant \dim M$.

Proof. First, we claim that we can reduce to the local case. By Theorem 5.51, the localization at P of a minimal injective resolution of M is also a minimal injective resolution of M_P, and thus

$$\operatorname{injdim}(M) \geqslant \operatorname{injdim}_{R_P}(M_P).$$

Moreover,

$$\dim(M) = \sup\{\dim(M_P) \mid P \in \operatorname{Spec} R\}.$$

Therefore, it suffices to show that for all prime ideals P,

$$\operatorname{injdim}_{R_P}(M_P) \geqslant \dim(M_P).$$

This reduces the problem to showing that for any nonzero finitely generated module M over any noetherian local ring $(R, \mathfrak{m})$,

$$\operatorname{injdim}(M) \geqslant \dim(M).$$

Let $d = \dim M$ and choose a minimal prime $P \in \operatorname{Supp}(M) = \operatorname{V}(\operatorname{ann}(M))$ such that $\dim(R/P) = d$. By Theorem 6.40 from Commutative Algebra I, $P \in \operatorname{Ass}(M)$, and thus $PR_P \in \operatorname{Ass}(M_P)$, so

$$\mu_0(P, M) = \dim_{\kappa(P)} \left(\operatorname{Hom}_{R_P}(\kappa(P), M_P)\right) \neq 0.$$

By Theorem 5.61, $\mu_d(\mathfrak{m}, M) \neq 0$, and thus $\operatorname{injdim}(M) \geqslant d$ by Theorem 5.62. □

Theorem 5.64. *Let $(R, \mathfrak{m})$ be a noetherian local ring and M a nonzero finitely generated R-module of finite injective dimension. Then* $\operatorname{injdim}(M) = \operatorname{depth}(R)$.

Proof. Let $\underline{x} = x_1, \ldots, x_n \in \mathfrak{m}$ be a maximal regular sequence on R, so that $n = \operatorname{depth}(R)$. Note that $\operatorname{pdim}(R/(x)) = n$, as $R/(x)$ is minimally generated by $\operatorname{kos}(\underline{x})$. By Exercise 17,

$$\operatorname{Ext}^n_R(R/(\underline{x}), M) \neq 0.$$

Since we can compute $\operatorname{Ext}^i_R(R/(\underline{x}), M)$ using an injective resolution of M, the fact that $\operatorname{Ext}^n_R(R/(\underline{x}), M) \neq 0$ implies that

$$\operatorname{injdim}(M) \geqslant n = \operatorname{depth}(R).$$

On the other hand, since $\underline{x}$ is a maximal regular sequence on R, we have $\operatorname{depth}(R/(\underline{x})) = 0$, and thus $\mathfrak{m} \in \operatorname{Ass}(R/(\underline{x}))$ by Lemma 2.5. Therefore, we an inclusion of $R/\mathfrak{m}$ into $R/(\underline{x})$, giving us a short exact sequence

$$0 \longrightarrow R/\mathfrak{m} \longrightarrow R/(\underline{x}) \longrightarrow C \longrightarrow 0.$$

Applying $\operatorname{Hom}_R(-, M)$ gives us a long exact sequence

$$\cdots \longrightarrow \operatorname{Ext}^i_R(R/(\underline{x}), M) \longrightarrow \operatorname{Ext}^i_R(R/\mathfrak{m}, M) \longrightarrow \operatorname{Ext}^{i+1}_R(C, M) \longrightarrow \cdots.$$

Consider $i = \operatorname{injdim}(M)$. By Theorem 5.62,

$$\operatorname{Ext}^{\operatorname{injdim}(M)}(R/\mathfrak{m}, M) \neq 0.$$

Moreover, note that $\operatorname{Ext}^j_R(N, M) = 0$ for all $j \geqslant \operatorname{injdim}(M)$ and all R-modules N, so in particular $\operatorname{Ext}^{\operatorname{injdim}(M)+1}_R(C, M) = 0$. This gives us a surjection

$$\operatorname{Ext}^{\operatorname{injdim}(M)}_R(R/(\underline{x}), M) \longrightarrow \underbrace{\operatorname{Ext}^{\operatorname{injdim}(M)}_R(R/\mathfrak{m}, M)}_{\neq 0} \longrightarrow 0.$$

Therefore,

$$\operatorname{Ext}^{\operatorname{injdim}(M)}_R(R/(\underline{x}), M) \neq 0.$$

Since we can compute this Ext module using a minimal free resolution for $R/(x)$, we conclude that

$$\operatorname{injdim}(M) \leqslant \operatorname{pdim}(R/(\underline{x})) = n = \operatorname{depth}(R). \qquad \square$$

Theorem 5.65. *Let $(R, \mathfrak{m}, k)$ be a noetherian local ring and M a finitely generated R-module. Then*

$$\operatorname{injdim}_R(M) = \operatorname{injdim}_{\widehat{R}}(\widehat{M}).$$

Proof. We follow the same strategy as in Theorem 3.54: $\operatorname{Ext}^i_R(k, M)$ is a finite-dimensional k-vector space, and thus by Exercise 45

$$\operatorname{Ext}^i_R(k, M) \otimes_R \widehat{R} \cong \operatorname{Ext}^i_{\widehat{R}}(k, \widehat{M}).$$

Then

$$\begin{aligned} \operatorname{injdim}(M) &= \sup\{i \mid \operatorname{Ext}^i_R(k, M) \neq 0\} && \text{by Theorem 5.62} \\ &= \sup\{i \mid \operatorname{Ext}^i_{\widehat{R}}(k, \widehat{M}) \neq 0\} \\ &= \operatorname{injdim}_{\widehat{R}}(\widehat{M}) && \text{by Theorem 5.62.} \quad \square \end{aligned}$$

Remark 5.66. Notice that the proof of Theorem 5.65 shows more: it proves that the Bass numbers with respect to the maximal ideal $\mathfrak{m}$, which are measured by $\operatorname{Ext}^i_R(k, M)$, are preserved by completion.

Exercise 55. Let $(R, \mathfrak{m})$ be a noetherian local ring. Show that if exists a nonzero finitely generated injective R-module, then $\dim(R) = 0$.

Exercise 56. Let R be any ring. Prove that an R-module E is injective if and only if $\operatorname{Ext}^1_R(R/I, E) = 0$ for all ideals I.

5.5 Global dimension revisited

Now that we know more about injective dimension, we can wrap up a few loose ends we left hanging when we discussed regular rings. Recall that we defined the global dimension of a ring R to be

$$\operatorname{gldim}(R) = \sup\{\operatorname{pdim}_R(M) \mid M \text{ is a finitely generated } R\text{-module}\}.$$

We are now going to show that in fact if every finitely generated R-module has finite projective dimension, and that projective dimension is uniformly bounded by some fixed integer n, then every R-module has projective dimension at most n. Therefore,

$$\operatorname{gldim}(R) = \sup\{\operatorname{pdim}_R(M) \mid M \text{ is an } R\text{-module}\}.$$

To do this, we will actually show more: we will relate projective and injective dimension, and show that

$$\operatorname{pdim}(M) \leqslant n \text{ for all } R\text{-modules } M \iff \operatorname{injdim}(M) \leqslant n \text{ for all } R\text{-modules } M.$$

First, we need a few auxiliary results.

Theorem 5.67. *Let R be a ring and let N be an R-module. If $\operatorname{Ext}^{n+1}_R(R/I, N) = 0$ for all ideals I, then $\operatorname{injdim}(N) \leqslant n$.*

Proof. Let

$$0 \longrightarrow N \longrightarrow E^0 \xrightarrow{\partial^0} E^1 \xrightarrow{\partial^1} \cdots$$

be an injective resolution of N. Set $C = \operatorname{im}(\partial^{n-1})$, and note that

$$0 \longrightarrow C = \operatorname{im}(\partial^{n-1}) \longrightarrow E^n \xrightarrow{\partial^n} E^{n+1} \xrightarrow{\partial^{n+1}} \cdots$$

is an injective resolution for C. Then using these injective resolutions for C and N to compute $\operatorname{Ext}^*_R(-, C)$ and $\operatorname{Ext}^*_R(-, N)$, for all ideals I we have

$$\operatorname{Ext}^1_R(R/I, C) = \operatorname{Ext}^{n+1}(R/I, N) = 0.$$

By Exercise 56, C must be injective. We conclude that

$$0 \longrightarrow N \longrightarrow E^0 \xrightarrow{\partial^0} E^1 \xrightarrow{\partial^1} \cdots \longrightarrow E^{n-1} \xrightarrow{\partial^{n-1}} C \longrightarrow 0$$

is an injective resolution for N, and thus $\operatorname{injdim}(N) \leqslant n$. □

We also need a projective dimension version of Theorem 5.67, which we leave as an exercise.

Exercise 57. Let M be an R-module. If $\operatorname{Ext}^i_R(M, N) = 0$ for all R-modules N and all $i > n$, then $\operatorname{pdim}(M) \leqslant n$.

Theorem 5.68. *Let R be any ring and $n \geqslant 0$ an integer. The following are equivalent:*

1) $\operatorname{pdim}(R/I) \leqslant n$ *for every proper ideal I of R.*

2) $\operatorname{injdim}(N) \leqslant n$ *for all R-modules N.*

3) $\operatorname{Ext}^i_R(M, N) = 0$ *for all $i > n$ and all R-modules M and N.*

4) $\operatorname{pdim}(M) \leqslant n$ *for every R-module M.*

Proof. 1 $\implies$ 2: Let N be an R-module and I be any proper ideal of R. By assumption, $\operatorname{pdim}(R/I) \leqslant n$, so using a projective resolution of R/I of length at most n, we conclude that

$$\operatorname{Ext}^{n+1}_R(R/I, N) = 0.$$

By Theorem 5.67, $\operatorname{injdim}(N) \leqslant n$.

2 $\implies$ 3 is immediate since we can take an injective resolution E for N of length $\operatorname{injdim}(N)$, and for all $i > n \geqslant \operatorname{injdim}(N)$ we have

$$\operatorname{Ext}^i_R(M, N) = \operatorname{H}^i(\operatorname{Hom}_R(M, E)) = 0.$$

3 $\implies$ 4 is Exercise 57.

4 $\implies$ 1 is immediate. □

Theorem 5.69. *Let $(R, \mathfrak{m}, k)$ be a noetherian local ring. Let $n \geqslant 0$ be an integer. The following conditions are equivalent:*

1) $\sup\{\operatorname{pdim}(M) \mid M$ *any R-module*$\} = n$.

2) $\operatorname{pdim}(k) = n$.

3) $\operatorname{injdim}(k) = n$.

4) $\sup\{\operatorname{injdim}(M) \mid M$ *any R-module*$\} = n$.

5) *The ring R has global dimension n.*

Proof. 2 $\iff$ 5: By Theorem 1.71, $\operatorname{pdim}(k) = n$ if and only if R is regular of dimension n. By Theorem 1.82, R is regular of dimension n if and only if $\operatorname{gldim}(R) = n$.

1 $\implies$ 2: if every R-module has projective dimension at most n, then so does k. On the other hand, by Corollary 1.24, $\operatorname{pdim}(R/I) \leqslant \operatorname{pdim}(k)$ for all ideals I. If $\operatorname{pdim}(k) < n$, then $\operatorname{pdim}(R/I) < n$ for all ideals I, and so by Theorem 5.68 we conclude that $\operatorname{pdim}(M) < n$ for all M.

2 $\implies$ 3: Since $\operatorname{pdim}(k) = n$, we conclude that $\operatorname{Ext}^i_R(k, k) = 0$ for all $i > n$. By Exercise 17, $\operatorname{Ext}^n_R(k, k) \neq 0$. Therefore, since by Theorem 5.62 we can compute the injective dimension via $\operatorname{Ext}^i_R(k, -)$, we conclude that $\operatorname{injdim}(k) = n$.

3 $\implies$ 4: Since $\operatorname{injdim}(k) = n$, $\operatorname{Ext}^n_R(k,k) \neq 0$ by Theorem 5.62. In particular, $\operatorname{pdim}(k) \geqslant n$. On the other hand, since k has $\operatorname{injdim}(k) = n$, then for all ideals I and all $i > n$ we have $\operatorname{Ext}^i_R(R/I, k) = 0$. By Remark 1.22,

$$\operatorname{pdim}(R/I) = \sup\{i \mid \operatorname{Ext}^i_R(R/I,k) \neq 0\},$$

and thus $\operatorname{pdim}(R/I) \leqslant n$ for all ideals I. By Theorem 5.68, $\operatorname{injdim}(M) \leqslant n$ for all M.

4 $\implies$ 1 is already captured by Theorem 5.68. Indeed, 4 implies $\operatorname{pdim}(M) \leqslant n$ for all M, but if $\operatorname{pdim}(M) < n$ for all R-modules M, then by Theorem 5.68 we must also have $\operatorname{injdim}(N) < n$ for all R-modules N. □

Note that as a consequence of Theorem 5.69, we obtain the following: if there exists any R-module with infinite projective dimension, then there is necessarily a finitely generated module (the residue field) of infinite projective dimension.

We can now upgrade Theorem 1.82:

Theorem 5.70. *Let R be a noetherian ring of finite dimension d. The ring R is a regular ring if and only if every R-module M has* $\operatorname{pdim}(M) \leqslant d$.

Proof. If every R-module has $\operatorname{pdim}(M) \leqslant d$, then in particular $\operatorname{gldim}(R) \leqslant d$, so R is regular by Theorem 1.82. Conversely, if R is a regular ring, then by Theorem 1.71 every cyclic module has finite projective dimension at most d, and by Theorem 5.68 every R-module M has $\operatorname{pdim}(M) \leqslant d$. □

While our original definition of global dimension only considered the projective dimension of finitely generated R-modules, Theorem 5.69 shows that the supremum does not change when taken over all R-modules. In particular, we can now update our definition to

$$\operatorname{gldim}(R) = \sup\{\operatorname{pdim}(M) \mid M \text{ any } R\text{-module}\}.$$

5.6 Matlis Duality

Eben Matlis proved [Mat58] a duality between noetherian and Artinian modules over complete noetherian rings $(R, \mathfrak{m})$ involving the functor $\mathrm{Hom}_R(-, E(k))$.

Definition 5.71. Let $(R, \mathfrak{m}, k)$ be a noetherian local ring. Set $E = E(k)$. The **Matlis functor** is the functor

$$(-)^\vee := \mathrm{Hom}_R(-, E) \colon R\textbf{-Mod} \longrightarrow R\textbf{-Mod}$$

given by

$$M^\vee = \mathrm{Hom}_R(M, E).$$

We saw that $M^\vee$ is the **Matlis dual** of M.

Remark 5.72. Note that since $E = E(k)$ is injective, the Matlis functor $(-)^\vee$ is exact.

Here is some motivation for studying the Matlis functor.

Remark 5.73. Let R be a noetherian ring. Any functor of the form $\mathrm{Hom}_R(-, E)$ with E an injective module is exact, and by Lemma 5.46, all such functors decompose as direct sums of $]\mathrm{Hom}_R(-, E)$ with $E = E(R/P)$ an indecomposable injective module. However, when $P \neq \mathfrak{m}$ we can always find modules that vanish under $\mathrm{Hom}_R(-, E(R/P))$: for example, we know that

$$\mathrm{Hom}_R(R/\mathfrak{m}, E(R/P)) = 0$$

by Lemma 5.45. Therefore, the only the only indecomposable injective E such that $\mathrm{Hom}_R(M, E)$ has a chance of being nonzero for all modules M is $E(k)$.

Lemma 5.74. *Let $(R, \mathfrak{m}, k)$ be a noetherian local ring, M any R-module, and $E = E(k)$. The natural map*

$$\begin{aligned} M &\longrightarrow M^{\vee\vee} = \mathrm{Hom}_R(\mathrm{Hom}_R(M, E), E) \\ m &\longmapsto \mathrm{ev}_m = (f \mapsto f(m)) \end{aligned}$$

is injective.

Proof. Fix a nonzero element $m \in M$. We want to show that the homomorphism

$$\begin{aligned} \mathrm{ev}_m \colon \mathrm{Hom}_R(M, E) &\longrightarrow E \\ f &\longmapsto f(m) \end{aligned}$$

is nonzero, so we need to prove that there exists $f \in \mathrm{Hom}_R(M, E)$ such that $f(m) \neq 0$.

Let $L = Rm \subseteq M$. Note that $L/\mathfrak{m}L$ is a cyclic k-module, so $L/\mathfrak{m}L \cong k$. Consider the homomorphism $\varphi \colon L \to E$ obtained by composing the canonical projection $L \longrightarrow L/\mathfrak{m}L \cong k$ with the inclusion $k \subseteq E(k)$. Note that m is a minimal generator of L, so $\varphi(m) \neq 0$.

Since E is injective, we can extend φ to a map $\tilde{\varphi}\colon M \longrightarrow E$, completing the commutative diagram

$$\begin{array}{ccccc} & & E & & \\ & & \uparrow{\scriptstyle \varphi} & \nwarrow{\scriptstyle \tilde{\varphi}} & \\ 0 & \longrightarrow & L & \longrightarrow & M \end{array}$$

In particular, note that $\tilde{\varphi}(m) = \varphi(m) \neq 0$. Thus we can take $f = \tilde{\varphi}$ above, proving that the double Matlis dual map $M \longrightarrow M^{\vee\vee}$ given by $m \mapsto \mathrm{ev}_m$ is injective. □

Remark 5.75. The use of the word *natural* in Lemma 5.74 is not colloquial: this is in fact natural in the formal sense, meaning that for all R-module homomorphisms $f\colon M \longrightarrow N$, the natural map induces a commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \downarrow & & \downarrow \\ M^{\vee\vee} & \xrightarrow[f^{\vee\vee}]{} & N^{\vee\vee} \end{array}$$

Here $f^{\vee\vee}\colon M^{\vee\vee} \longrightarrow N^{\vee\vee}$ sends each $g\colon \mathrm{Hom}_R(M,E) \longrightarrow E$ to $g(-\circ f)\colon \mathrm{Hom}_R(N,E) \longrightarrow E$, that is, $f^{\vee\vee}(g) \in N^{\vee\vee} = \mathrm{Hom}_R(\mathrm{Hom}_R(N,E),E)$ is the composition map

$$\begin{array}{ccccc} \mathrm{Hom}_R(N,E) & \longrightarrow & \mathrm{Hom}_R(M,E) & \longrightarrow & E \\ h & \longmapsto & h\circ f & \longmapsto & g(h\circ f). \end{array}$$

To check the diagram above commutes, fix $m \in M$, and write ev_m for its image in $M^{\vee\vee}$ under the natural map, which is takes each $f \in \mathrm{Hom}_R(M,E)$ to $\mathrm{ev}_m(f) = f(m) \in E$. Then $f^{\vee\vee}(\mathrm{ev}_m)$ is the composition

$$\begin{array}{ccccc} \mathrm{Hom}_R(N,E) & \longrightarrow & \mathrm{Hom}_R(M,E) & \longrightarrow & E \\ h & \longmapsto & h\circ f & \longmapsto & \mathrm{ev}_m(h\circ f) \end{array}$$

and

$$\mathrm{ev}_m(h\circ f) = (h\circ f)(m) = h(f(m)).$$

Therefore, $f^{\vee\vee}(\mathrm{ev}_m)$ is the map given by

$$h \mapsto h(f(m)).$$

Now following the diagram the other direction, the natural map sends $f(m)$ to the element

$$\mathrm{ev}_{f(m)} \in N^{\vee\vee} = \mathrm{Hom}_R(\mathrm{Hom}_R(N,E),E),$$

which is the map that sends each $h \in \mathrm{Hom}_R(N,E)$ to

$$h(f(m)).$$

This shows that the diagram does indeed commute, and thus the assignment $M \mapsto M^{\vee\vee}$ is a natural transformation.

Corollary 5.76. *Let $(R, \mathfrak{m}, k)$ be a noetherian local ring and M be any nonzero R-module. Then $\mathrm{Hom}_R(M, E(k)) = M^\vee \neq 0$.*

Proof. If $M \neq 0$, then $M^{\vee\vee} \neq 0$ by Lemma 5.74, and thus

$$\mathrm{Hom}_R(M, E(R/\mathfrak{m})) = M^\vee \neq 0. \qquad \square$$

Lemma 5.77. *Let $(R, \mathfrak{m}, k)$ be a noetherian local ring and M be an R-module of finite length. Let $E = E(k)$. Then $M^\vee$ also has finite length, and in fact*

$$\ell(M) = \ell(M^\vee).$$

Proof. We proceed by induction on $\ell(M)$. When $\ell(M) = 1$, $M \cong R/\mathfrak{m}$, so

$$\begin{aligned} \mathrm{Hom}_R(M, E) &\cong \mathrm{Hom}_R(R/\mathfrak{m}, E) \\ &\cong E_{R/\mathfrak{m}}(M) && \text{by Lemma 5.45} \\ &\cong R/\mathfrak{m} && \text{by Lemma 5.36} \end{aligned}$$

Therefore,

$$\ell(M^\vee) = \ell(\mathrm{Hom}_R(M, E)) = \ell(R/\mathfrak{m}) = 1.$$

Now assume M is a module of finite length $\ell(M) = n \geqslant 2$ and that the result holds for all R-modules of length less than n. Since M has finite length, $\mathfrak{m} \in \mathrm{Ass}(M)$ and we can find a submodule of M isomorphic to R/m. This gives us a short exact sequence

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

with $\ell(L) = 1$ and $\ell(N) = n - 1$. Since E is injective, $\mathrm{Hom}_R(-, E)$ is an exact functor. Applying $\mathrm{Hom}_R(-, E)$ to the short exact sequence above, we obtain the exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(N, E) \longrightarrow \mathrm{Hom}_R(M, E) \longrightarrow \mathrm{Hom}_R(L, E) \longrightarrow 0$$

We have shown that $\mathrm{Hom}_R(L, E) = 1$. By induction hypothesis,

$$\ell(\mathrm{Hom}_R(N, E)) = \ell(N) = n - 1.$$

By the additivity of length on short exact sequences from Lemma 9.6 from Commutative Algebra I, we conclude that

$$\ell(M^\vee) = \ell(\mathrm{Hom}_R(M, E)) = \ell(\mathrm{Hom}_R(N, E)) + \ell(\mathrm{Hom}_R(L, E)) = 1 + (n - 1) = \ell(M). \qquad \square$$

Corollary 5.78. *Let $(R, \mathfrak{m}, k)$ be a noetherian local ring and M be an R-module of finite length. There is a natural isomorphism $M^{\vee\vee} \cong M$.*

Proof. By Lemma 5.74, there is a natural inclusion of M into $M^{\vee\vee}$. By Lemma 5.77,

$$\ell(M) = \ell(M^\vee) = \ell(M^{\vee\vee}).$$

Therefore, the inclusion $M \subseteq M^{\vee\vee}$ must be an isomorphism. $\square$

Remark 5.79. Let $(R, \mathfrak{m}, k)$ be a noetherian local ring and $\widehat{R}$ its completion with respect to the maximal ideal $\mathfrak{m}$. Let M be any R-module that is $\mathfrak{m}$-torsion, meaning that every element in M is killed by some power of $\mathfrak{m}$. Then M is also an $\widehat{R}$-module, as follows. Given $x \in M$ and $\hat{r} \in \widehat{R}$, fix n such that $\mathfrak{m}^n x = 0$.

Thinking of $\hat{r}$ as a consistent sequence

$$\hat{r} = \{r_e + \mathfrak{m}^e\} \in \prod_e R/\mathfrak{m}^e,$$

meaning that $r_{e+1} - r_e \in \mathfrak{m}^e$ for all e, set $r = r_n$, and note that $r_e - r_n \in \mathfrak{m}^n$ for all $e \geqslant n$. We set

$$\hat{r}x := rx.$$

Since $\mathfrak{m}^n x = 0$, this is well-defined: any other choice of $s \in R$ such that $s - r \in \mathfrak{m}^n$ would satisfy

$$(s-r)x \in \mathfrak{m}^n\, x = 0 \implies sx = rx.$$

We leave it as an exercise to check that this does in fact make M an $\widehat{R}$-module.

Theorem 5.80. *Let $(R, \mathfrak{m}, k)$ be a noetherian local ring. Then*

$$E_{\widehat{R}}(k) = E_R(k).$$

Proof. By Corollary 5.35, $E_R(k)$ is $\mathfrak{m}$-torsion, so it inherits the structure of a $\widehat{R}$-module by Remark 5.79. Note in fact that all submodules of $E_R(k)$ are $\mathfrak{m}$-torsion, so its R-submodules and $\widehat{R}$-submodules coincide. Thus $k \subseteq E_R(k)$ is an essential extension both as R-modules and $\widehat{R}$-modules.

Now suppose that $E_R(k) \subseteq M$ is an essential extension of $\widehat{R}$-modules. By restriction of scalars, M is also an R-module. By Theorem 5.9 and Corollary 5.31,

$$\{\mathfrak{m}\} = \operatorname{Ass}_R(E_R(k)) = \operatorname{Ass}_R(M).$$

By Lemma 5.34, every element of M is $\mathfrak{m}$-torsion. The $\widehat{R}$-module structure of M must then follow the construction in Remark 5.79, and thus for all $\hat{r} \in \widehat{R}$ and all $x \in M$ there exists $r \in R$ such that $\hat{r}x = rx$. Therefore, $\widehat{R}x = R\,x$. Given any nonzero $x \in M$, we now see that since $E_R(k) \subseteq M$ is an essential extension of $\widehat{R}$-modules,

$$R\,x \cap E_R(k) = \widehat{R}\,x \cap E_R(k) \neq 0.$$

Therefore, $E_R(k) \subseteq M$ is an essential extension of R-modules. But $E_R(k)$ is injective as an R-module, so by Theorem 5.17 we conclude that $E_R(k) = M$. Thus $k \subseteq E_R(k)$ is a maximal essential extension of $\widehat{R}$-modules, and we conclude that

$$E_{\widehat{R}}(k) = E_R(k). \qquad \square$$

Remark 5.81. Let $(R, \mathfrak{m}, k)$ be a noetherian local ring and $E = E(R/\mathfrak{m})$. For each n, set $R_n := R/\mathfrak{m}^n$ and

$$E_n := \{e \in E \mid \mathfrak{m}^n e = 0\}.$$

Every R_n-module A can be seen as an R-module, defining

$$ra := (r + \mathfrak{m}^n)\, a$$

for each $r \in R$ and $a \in A$. In particular, if $A \subseteq B$ is an essential extension of R_n-modules, it is also an essential extension of R-modules, since B is annihilated by $\mathfrak{m}^n$, and therefore the R-submodules and R_n-submodules of B coincide.

Theorem 5.82. *Let $(R, \mathfrak{m}, k)$ be a noetherian local ring. Set $E = E_R(k)$. For all $n \geqslant 1$,*

$$E_{R/\mathfrak{m}^n}(R/\mathfrak{m}) = \operatorname{Hom}_R(R/\mathfrak{m}^n, E) \cong (0 :_E \mathfrak{m}^n).$$

Proof. Apply Lemma 5.44 to $I = \mathfrak{m}^n$. □

Theorem 5.83. *Let $(R, \mathfrak{m}, k)$ be a noetherian local ring. Then*

$$E_R(R/\mathfrak{m}) = \bigcup_n E_{R/\mathfrak{m}^n}(R/\mathfrak{m}).$$

Proof. Set $R_n := R/\mathfrak{m}^n$ and

$$E_n := \{e \in E \mid \mathfrak{m}^n e = 0\}.$$

Since each E_n is by construction an submodule of E, all we have to prove is that

$$E \subseteq \bigcup_n E_n.$$

Let $x \in E$, and consider the submodules of E of the form $\mathfrak{m}^n x \subseteq (x)$. Since R is noetherian, and (x) is a finitely generated R-module, then by Krull's Intersection Theorem, see **??** from Commutative Algebra I,

$$\bigcap_n \mathfrak{m}^n = 0.$$

Therefore,

$$\left(\bigcap_n \mathfrak{m}^n\right) x = \bigcap_n \mathfrak{m}^n\, x = 0.$$

Suppose $\mathfrak{m}^n\, x \neq 0$ for all n. Then for each n, $\mathfrak{m}^n\, x$ is a nonzero submodule of E, which is an essential extension of $R/\mathfrak{m}$, and therefore $R/\mathfrak{m} \cap \mathfrak{m}^n\, x \neq 0$. Since $R/\mathfrak{m}$ is a simple module, and $0 \neq R/\mathfrak{m} \cap \mathfrak{m}^n\, x \subseteq R/\mathfrak{m}$, we must have

$$R/\mathfrak{m} \cap \mathfrak{m}^n\, x = R/\mathfrak{m}.$$

Thus

$$R/\mathfrak{m} \subseteq \bigcap_n \mathfrak{m}^n\, x = 0,$$

which is impossible because $R/\mathfrak{m} \neq 0$. Then $\mathfrak{m}^n\, x = 0$ for some n, and by definition, this means that $x \in E_n$. □

Theorem 5.84. *Let $(R, \mathfrak{m}, k)$ be a noetherian local ring and $E = E(k)$. There is an isomorphism of R-modules*

$$\begin{aligned} \widehat{R} &\longrightarrow R^{\vee\vee} = \mathrm{Hom}_R(E, E) \\ r &\longmapsto (e \mapsto re). \end{aligned}$$

Proof. First, note that indeed

$$R^{\vee\vee} = \mathrm{Hom}_R(\mathrm{Hom}_R(R, E), E) \cong \mathrm{Hom}_R(E, E).$$

By Corollary 5.35, $E_R(k)$ is $\mathfrak{m}$-torsion, and by Theorem 5.80, $E_R(k) \cong E_{\widehat{R}}(k)$, so any homomorphism of R-modules $E \longrightarrow E$ is also a homomorphism of $\widehat{R}$-modules. Thus

$$\mathrm{Hom}_R(E, E) = \mathrm{Hom}_{\widehat{R}}(E, E).$$

We can now assume that R is complete, and need only to show that $R \cong E^{\vee\vee}$. We claim that the map above is simply the natural map $R \mapsto R^{\vee\vee}$. First, note that the natural isomorphism $\mathrm{Hom}_R(R, E) \cong E$ consists simply of identifying each homomorphism with the element $e \in E$ that $1 \in R$ maps to. Thus the natural map $R \cong E^{\vee\vee}$ sends $r \in R$ to the map in $R^{\vee\vee}$ that sends each $e \in R^{\vee}$ to $e(r) = re$.

We showed in Lemma 5.74 that the natural map is injective, so all that remains is to show that this map $R \mapsto \widehat{R}$ is surjective. When R is artinian, the length $\ell(R)$ is finite, and by Corollary 5.78 the natural map $R \longrightarrow R^{\vee\vee}$ is an isomorphism.

Now consider any noetherian local ring R. For each n, set $E_n = (0 :_E \mathfrak{m}^n)$. For each $\varphi \in \mathrm{Hom}_R(E, E)$, by R-linearity

$$\mathfrak{m}^n \varphi(E_n) = \varphi(\mathfrak{m}^n E_n) = \varphi(0) = 0.$$

Therefore, $\varphi(E_n) \subseteq E_n$, and thus φ restricts to an element $\varphi_n \in \mathrm{Hom}_{R/\mathfrak{m}^n}(E_n, E_n)$. By Theorem 5.82, $E_n = E_{R/m^n}(k)$. On the other hand, $R/\mathfrak{m}^n$ is an artinian ring, so the natural map

$$R/\mathfrak{m}^n \longrightarrow \mathrm{Hom}_{R/\mathfrak{m}^n}(E_n, E_n)$$

is surjective. Therefore, the restriction of φ to $\varphi_n \in \mathrm{Hom}_{R/\mathfrak{m}^n}(E_n, E_n)$ is given by multiplication by an element $r_n \in R/\mathfrak{m}^n$. Note moreover, that for $e > n$, φ_e restricts to φ_n, so these elements r_n are consistent with each other:

$$r_e - r_n \in \mathfrak{m}^n.$$

Since this holds for all n and by Theorem 5.83

$$E_R(R/\mathfrak{m}) = \bigcup_n E_{R/\mathfrak{m}^n}(R/\mathfrak{m})$$

we conclude that $\varphi \in \mathrm{Hom}_R(E, E)$ is given by multiplication by $r = \{r_n\} \in \widehat{R} = R$. □

Corollary 5.85. *Let $(R, \mathfrak{m}, k)$ be a noetherian local ring and $E = E(k)$. Then $E^\vee \cong \widehat{R}$.*

Proof. Since $R^\vee = \mathrm{Hom}_R(R, E) \cong E$, then by Theorem 5.84

$$E^\vee = R^{\vee\vee} \cong \widehat{R}.$$ □

Theorem 5.86. *For any noetherian local ring $(R, \mathfrak{m}, k)$, $E(k)$ is an artinian R-module.*

Proof. Consider any descending chain of submodules of $E = E(k)$, say

$$E \supseteq E_1 \supseteq E_2 \supseteq E_3 \supseteq \cdots.$$

Applying the Matlis functor, which is exact and thus takes inclusions into surjections, we get a sequence of surjections

$$E^\vee = \widehat{R} \twoheadrightarrow E_1^\vee \twoheadrightarrow E_2^\vee \twoheadrightarrow \cdots.$$

Consider the ideals of $\widehat{R}$ given by $I_i = \ker(\widehat{R} \twoheadrightarrow E_i^\vee)$, and note that they form an ascending chain

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots.$$

Since R is a noetherian ring, its completion $\widehat{R}$ is also noetherian, by Theorem 3.42. Therefore, this ascending chain of ideals of $\widehat{R}$ stabilizes; let N be such that $I_n = I_N$ for all $n \geqslant N$. Then the quotients $E_n^\vee \twoheadrightarrow E_N^\vee$ must all be isomorphisms for $n \geqslant N$.

We claim that $E_n = E_N$ for all $n \geqslant N$. Suppose not, and fix n such that $E_n \neq E_N$, so that $Q = E_n/E_N \neq 0$. Note that applying the (exact) Matlis functor to the short exact sequence

$$0 \longrightarrow E_N \longrightarrow E_n \longrightarrow Q \longrightarrow 0$$

gives

$$0 \longrightarrow Q^\vee \longrightarrow E_N^\vee \xrightarrow{\cong} E_n^\vee \longrightarrow 0$$

and thus $Q^\vee = 0$. But $Q \neq 0$ implies $Q^\vee \neq 0$ by Corollary 5.76, which is a contradiction. We conclude that $E_n = E_N$ for all $n \geqslant N$, and thus the original descending chain of submodules of E stops. We conclude that E is an artinian R-module. □

Definition 5.87. Let M be a module over a noetherian local ring $(R, \mathfrak{m})$. The **socle** of M is the submodule

$$\mathrm{soc}(M) := (0 :_M \mathfrak{m}) = \mathrm{Hom}_R(R/\mathfrak{m}, M).$$

Remark 5.88. Let $(R, \mathfrak{m}, k)$ be a noetherian local ring, and let M be an R-module with $\mathfrak{m} \in \mathrm{Ass}(M)$. Then we can find a nonzero element $x \in M$ such that $\mathfrak{m}x = 0$, so $x \in \mathrm{soc}(M)$. Therefore, whenever $\mathfrak{m} \in \mathrm{Ass}(M)$, $\mathrm{soc}(M)$ is nonzero. Conversely, if $\mathrm{soc}(M)$ is nonzero, then there exists an element $x \in M$ such that $\mathrm{ann}(x) = \mathfrak{m}$, so $\mathfrak{m} \in \mathrm{Ass}(M)$.

Lemma 5.89. *Let $(R, \mathfrak{m}, k)$ be a noetherian local ring and N a nonzero artinian R-module. Then the inclusion $\operatorname{soc}(N) \subseteq N$ is an essential extension.*

Proof. Given any nonzero $x \in N$, note that Rx is a submodule of an artinian module, by Theorem 5.86, and thus artinian by Exercise 23 from Commutative Algebra I, but also finitely generated, and thus noetherian. By Lemma 9.11 from Commutative Algebra I, Rx is a finite length module. By Exercise 24 from Commutative Algebra I, there exists n such that $\mathfrak{m}^n x = 0$. Taking n smallest possible, which must necessarily be positive since $x \neq 0$, note that $0 \neq \mathfrak{m}^{n-1} x \subseteq \operatorname{soc}(N)$. This shows that $Rx \cap \operatorname{soc}(N) \neq 0$, and thus $\operatorname{soc}(N) \subseteq N$ is an essential extension. □

Theorem 5.90. *Let $(R, \mathfrak{m}, k)$ be a noetherian local ring and M an artinian R-module. Then M can be embedded into a direct sum of a finite number of copies of $E(R/\mathfrak{m})$.*

Proof. Let M be any artinian R-module. By Lemma 9.13 from Commutative Algebra I, $\operatorname{Ass}(N) = \{\mathfrak{m}\}$. By Theorem 5.30, $\operatorname{Ass}(E(N)) = \operatorname{Ass}(N) = \{\mathfrak{m}\}$. By Corollary 5.43, $E(N)$ is a direct sum of copies of $E(R/\mathfrak{m})$, say $E(N) = E(R/\mathfrak{m})^{\alpha}$.

Now consider the socle submodule of N,

$$\operatorname{soc}(N) = (0 :_N \mathfrak{m}) = \operatorname{Hom}_R(R/\mathfrak{m}, N).$$

By Lemma 5.89, $\operatorname{soc}(N) \subseteq N$ is an essential extension.

Consider the inclusions $\operatorname{soc}(N) \subseteq N \subseteq E(N)$. By Theorem 5.8, $\operatorname{soc}(N) \subseteq E(N)$ is also an essential extension. Since $E(N)$ is injective, by Theorem 5.17 we conclude that

$$E(\operatorname{soc}(N)) = E(N).$$

We claim that $\operatorname{soc}(N)$ is a finitely generated R-module. By definition, $\operatorname{ann}(\operatorname{soc}(N)) = \mathfrak{m}$, and thus $\operatorname{soc}(N)$ is a vector space over $R/\mathfrak{m}$. On the other hand, $\operatorname{soc}(N)$ is also artinian, since it is a submodule of the artinian module N. Over a field, artinian modules must also be noetherian. We conclude that $\operatorname{soc}(N)$ is a finite-dimensional vector space over $R/\mathfrak{m}$, and thus a finitely generated R-module. By Lemma 5.55, $E(N) = E(\operatorname{soc}(N))$ is a direct sum of finitely many copies of $E(R/\mathfrak{m})$. □

Theorem 5.91. *Let $(R, \mathfrak{m}, k)$ be a noetherian local ring and M a finitely generated R-module. Then $M^\vee$ is an artinian R-module.*

Proof. Since R is a noetherian ring, we can find a finite presentation for M, say

$$R^n \longrightarrow R^m \longrightarrow M \longrightarrow 0$$

and since the Matlis functor is exact and $R^\vee = \operatorname{Hom}_R(R, E) \cong E$, we get an exact sequence

$$0 \longrightarrow M^\vee \longrightarrow E^m \longrightarrow E^n.$$

By Theorem 5.86, E is an artinian R-module, and thus E^m and any of its submodules must also be artinian, by Exercise 23 from Commutative Algebra I. We conclude that $M^\vee$ is artinian. □

Theorem 5.92 (Matlis Duality)**.** *Let $(R, \mathfrak{m}, k)$ be a complete noetherian local ring and $E = E(k)$. Denote the Matlis functor by $(-)^\vee = \mathrm{Hom}_R(-, E)$.*

1) *If M is any noetherian R-module, then $M^\vee$ is artinian and the natural map $M \longrightarrow M^{\vee\vee}$ is an isomorphism.*

2) *If N is any artinian R-module, then $N^\vee$ is noetherian and the natural map $N \longrightarrow N^{\vee\vee}$ is an isomorphism.*

Proof. Fix a noetherian R-module M and an artinian R-module N.

First, we look at R and E. We know that $R^\vee = \mathrm{Hom}_R(R, E) \cong E$, and by Theorem 5.84 the natural map $R = \widehat{R} \longrightarrow R^{\vee\vee}$ is an isomorphism, so $R^{\vee\vee} \cong R$.

By Corollary 5.85, we know that $E^\vee \cong R$. Thus

$$E^{\vee\vee} \cong R^\vee \cong E.$$

As in the proof of Theorem 5.91, M has a finite presentation, say

$$R^n \longrightarrow R^m \longrightarrow M \longrightarrow 0$$

and since the Matlis functor is exact and $R^\vee \cong E$, we get an exact sequence

$$0 \longrightarrow M^\vee \longrightarrow E^m \longrightarrow E^n.$$

Now apply the Matlis functor again. By Theorem 5.84, the natural map $R \longrightarrow R^{\vee\vee}$ is an isomorphism. We get a commutative diagram

$$\begin{array}{ccccccc} R^n & \longrightarrow & R^m & \longrightarrow & M & \longrightarrow & 0 \\ \downarrow \cong & & \downarrow \cong & & \downarrow & & \\ R^n & \longrightarrow & R^m & \longrightarrow & M^{\vee\vee} & \longrightarrow & 0 \end{array}$$

By the Five Lemma, we conclude that the map $M \longrightarrow M^{\vee\vee}$ is an isomorphism.

Now consider a nonzero artinian R-module N. By Theorem 5.90, the injective hull of N is a finite direct sum of copies of $E(R/\mathfrak{m})$, say $E(N) = E(R/\mathfrak{m})^\alpha$. By Theorem 5.86, this is a direct sum of artinian modules, and thus $E(N)$ is an artinian module. Therefore, the cokernel of the inclusion $N \subseteq E(N)$ is a quotient of an artinian module, which must be artinian by Exercise 23 from Commutative Algebra I. Again by Theorem 5.90, its injective hull must be a finite direct sum of copies of E, say E^β. Following the procedure to form an injective resolution for N that we described in Theorem 5.50, we get an exact sequence

$$0 \longrightarrow N \longrightarrow E^\alpha \longrightarrow E^\beta.$$

Applying the Matlis functor, which is exact, we get an exact sequence

$$R^\beta \longrightarrow R^\alpha \longrightarrow N^\vee \longrightarrow 0.$$

This is a finite presentation for $N^\vee$, and therefore $N^\vee$ is a noetherian R-module since R is noetherian.

Applying the Matlis functor again, we get a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & M & \longrightarrow & E^\alpha & \longrightarrow & E^\beta \\ & & \downarrow & & \downarrow\cong & & \downarrow\cong \\ 0 & \longrightarrow & M^{\vee\vee} & \longrightarrow & E^\alpha & \longrightarrow & E^\beta \end{array}$$

and by the Five Lemma we conclude that the map $M \longrightarrow M^{\vee\vee}$ must be an isomorphism. □

Recall that the an R-module M has finite length if and only if M is both noetherian and artinian. Hence the category of finite length R-modules is a subcategory of both the categories of noetherian and artinian R-modules.

Corollary 5.93. *Let $(R, \mathfrak{m}, k)$ be a complete noetherian local ring. The Matlis functor establishes a contravariant equivalence of categories between the category of noetherian R-modules and the category of artinian R-modules. Moreover, this functor restrics to a contravariant autoequivalence of categories on the subatogery of finite length R-modules.*

Proof. Let $\mathcal{A}(R)$ denote the category of artinian R-modules, and $\mathcal{N}(R) = R\textbf{-mod}$ the category of noetherian (aka finitely generated) R-modules. By Theorem 5.92, the restrictions of the Matlis functor to $\mathcal{N}(R)$ and $\mathcal{A}(R)$ give us functors

$$S \colon \mathcal{A}(R) \longrightarrow \mathcal{N}(R) \qquad T \colon \mathcal{N}(R) \longrightarrow \mathcal{A}(R)$$

where $S(N) = N^\vee$ and $T(M) = M^\vee$. By Theorem 5.92, there are natural isomorphisms between ST and the identity and TS and the identity, so S and T are equivalences of categories.

Moreover, recall that an R-module has finite length if and only if it is both artinian and noetherian, so $\mathcal{N} \cap \mathcal{A}$ is precisely the subcategory of finite length modules. By Corollary 5.78, the Matlis functor restricts further to a functor from finite length modules to finite length modules, which is an equivalence of categories. □

Exercises

Exercise 58. Let $(R, \mathfrak{m}, k)$ be an artinian local ring. Show that $E(k)$ is a finitely generated R-module.

Exercise 59. Let $(R, \mathfrak{m}, k)$ be a noetherian local ring.

1) Show that $\operatorname{ann}(E(k)) = 0$.

2) Give an example of such a ring R and an injective R-module I such that $\operatorname{ann}(I) \neq 0$.

Exercise 60. Let R be a noetherian local ring and E an injective R-module. Show that for all proper ideals I, $\operatorname{Hom}_R(R/I, E)$ is an injective module over R/I.

Exercise 61. Let

$$0 \longrightarrow E_1 \longrightarrow \cdots \longrightarrow E_n \longrightarrow 0$$

be an exact sequence of R-modules with $n \geqslant 3$. Show that if $E_1, \ldots, E_{n-1}$ are all injective, then so is E_n.

Exercise 62. Fix a proper ideal I, $n \geqslant 1$, and let M be an R-module such that $\operatorname{Ext}^i_R(R/I, M) = 0$ for all $i \neq n$. Show that if

$$0 \longrightarrow E^0 \xrightarrow{\partial^0} E^1 \longrightarrow \cdots$$

is an injective resolution for M, then

$$0 \longrightarrow \operatorname{coker}(\partial^{n-1}_*) \xrightarrow{\partial^n_*} \operatorname{Hom}_R(R/I, E^{n+1}) \xrightarrow{\partial^{n+1}_*} \operatorname{Hom}_R(R/I, E^{n+2}) \xrightarrow{\partial^{n+2}_*} \cdots$$

is an injective resolution for $\operatorname{Ext}^n_R(R/I, M)$, where $\partial^i_* = \operatorname{Hom}_R(R/I, \partial^i)$.

Chapter 6

Gorenstein rings

The study of injective modules and injective dimension leads to Gorenstein rings, which were first defined by Alexander Grothendieck ([Gro58]). Gorenstein rings are so named because of their relation to a duality property of singular plane curves studied by Daniel Gorenstein ([Gor52]), although Gorenstein himself claimed not to understand Gorenstein rings. Gorenstein rings are a special subclass of Cohen-Macaulay rings, and play an important role in several subfields across Commutative Algebra.

6.1 Gorenstein rings

Definition 6.1. A noetherian local ring R is a **Gorenstein ring** if R has finite injective dimension as an R-module.

Theorem 6.2. *Gorenstein local rings are Cohen-Macaulay.*

Proof. Let $(R, \mathfrak{m})$ be noetherian local ring that is Gorenstein. Then

$$\begin{aligned} \operatorname{depth}(R) &= \operatorname{injdim}(R) && \text{by Theorem 5.64} \\ &= \dim(R) && \text{by Corollary 5.63.} \end{aligned}$$

Since $\operatorname{depth}(R) \leqslant \dim(R)$ always holds, we conclude that $\dim(R) = \operatorname{depth}(R)$ and R is Cohen-Macaulay. □

The proof also shows the following:

Corollary 6.3. *If R is a Gorenstein local ring, then* $\operatorname{injdim}(R) = \dim(R)$.

Theorem 6.4. *All regular local rings are Gorenstein.*

Proof. Let $(R, \mathfrak{m}, k)$ be a regular local ring. By Auslander–Buchsbaum–Serre, Theorem 1.71, $\operatorname{pdim}(k) = \dim(R) < \infty$. By Theorem 5.69, every R-module M satisfies

$$\operatorname{injdim}(M) \leqslant \dim(R),$$

and thus in particular $\operatorname{injdim}(R) \leqslant \dim(R)$ is finite. □

Exercise 63. Let $(R, \mathfrak{m}, k)$ be a noetherian local ring. Show that R is an indecomposable R-module.

Exercise 64. Let $(R, \mathfrak{m}, k)$ be a noetherian local ring. Show that if R is injective then $R \cong E(k)$.

Artinian local rings are Gorenstein precisely when their socle dimension is 1.

Theorem 6.5. *Let $(R, \mathfrak{m}, k)$ be an artinian local ring. Then R is Gorenstein if and only if*

$$\operatorname{rank}_k(\operatorname{soc}(R)) = 1.$$

Proof. Artinian rings have dimension 0. If R is Gorenstein, then

$$\begin{aligned} \operatorname{depth}(R) &= \operatorname{injdim}(R) && \text{by Theorem 5.64} \\ &= \dim(R) && \text{by Corollary 5.63} \\ &= 0. \end{aligned}$$

Therefore, R is an injective R-module. By Exercise 64, $R \cong E(k)$. Therefore,

$$\begin{aligned} \operatorname{soc}(R) &= \operatorname{soc}(E(k)) \\ &= \operatorname{Hom}_R(k, E(k)) && \text{by definition of socle} \\ &= k && \text{by Lemma 5.45.} \end{aligned}$$

Thus

$$\operatorname{rank}_k(\operatorname{soc}(R)) = 1.$$

Now suppose that R has

$$\operatorname{rank}_k(\operatorname{soc}(R)) = 1.$$

Then $\operatorname{soc}(R) \cong k$, and since $E(k)$ is an injective module, there exists h such that

$$\begin{array}{ccccc} & & E(k) & & \\ & & \uparrow{\scriptstyle i} & \nwarrow{\scriptstyle h} & \\ 0 & \longrightarrow & k \cong \operatorname{soc}(R) & \xrightarrow[j]{} & R \end{array}$$

commutes.

Since R is an artinian R-module, then by Lemma 5.89 the inclusion $\operatorname{soc}(R) \subseteq R$ is an essential extension. Since $i = h \circ j$ and j are injective, then

$$\ker(h) \cap \operatorname{soc}(R) = \ker(h) \cap j(\operatorname{soc}(R)) = 0.$$

Since $j \colon \operatorname{soc}(R) \subseteq R$ is an essential extension, we conclude that h must be injective.

Now h is an inclusion $R \subseteq E(k)$, and R is a finite length R-module. By Corollary 5.78, $\ell(R) = \ell(E(k))$, and thus h must be an isomorphism. In particular, $R \cong E(k)$ is an injective module. □

Exercise 65. Let k be any field. For each of the following rings, determine which ones are Gorenstein:

1) $A = k[\![x]\!]/(x^3)$.

2) $B = k[\![x, y]\!]/(x^2, y^3)$.

3) $C = k[\![x, y]\!]/(x^2, xy, y^2)$.

4) $D = \mathbb{Z}/(4)$.

One of the first natural questions we may ask is whether the Gorenstein property localizes; the answer is, unsurprisingly, yes.

Theorem 6.6. *Let $(R, \mathfrak{m}, k)$ be a noetherian local ring. If R is Gorenstein, then R_P is Gorenstein for all primes P.*

Proof. By Theorem 5.51, injective resolutions localize, and thus

$$\operatorname{injdim}_R(R) \geqslant \operatorname{injdim}_{R_P}(R_P).$$

Thus if R has finite injective dimension as an R-module, then R_P has finite injective dimension over R_P. □

Definition 6.7. A noetherian ring R is **Gorenstein** if R_P is a Gorenstein local ring for all prime ideals P.

The following is a very useful fact:

Theorem 6.8. *Let M and N be R-modules and $x \in R$ be such that $xM = 0$ and x is regular on both R and N. Then*

$$\operatorname{Hom}_R(M, N) = 0$$

and for all $n \geqslant 0$ there is a natural isomorphism

$$\operatorname{Ext}^{n+1}_R(M, N) \cong \operatorname{Ext}^n_{R/(x)}(M, N/(x)N).$$

Proof. Given any $f \in \operatorname{Hom}_R(M, N)$, since $xM = 0$ then for all $m \in M$ we have

$$xf(m) = f(xm) = f(0) = 0.$$

Since x is regular on N, we conclude that $f(m) = 0$. Therefore, $\operatorname{Hom}_R(M, N) = 0$.

To show that there is a natural isomorphism $\operatorname{Ext}^{n+1}_R(M, N)$ and $\operatorname{Ext}^n_{R/(x)}(M, N/(x)N)$ for all $n \geqslant 0$, by Theorem 6.7 from Homological Algebra we need only to show that the functors

$$F^n = \operatorname{Ext}^{n+1}_R(-, N)$$

send $R/(x)$-modules to $R/(x)$-modules, and satisfy the following properties:

1) $F^0(-)$ is naturally isomorphic to $\mathrm{Hom}_{R/(x)}(-, N/(x)N)$.
2) $F^n(P) = 0$ for all $n > 0$ and all projective modules P over $R/(x)$.
3) For any short exact sequence of modules over $R/(x)$,

$$0 \longrightarrow A \longrightarrow B \longrightarrow B \longrightarrow C \longrightarrow 0$$

there is a long exact sequence (which is natural in the category of short exact sequences of modules over $R/(x)$)

$$\cdots \longrightarrow F^i(C) \longrightarrow F^i(B) \longrightarrow F^i(A) \longrightarrow F^{i+1}(C) \longrightarrow \cdots$$

First, note that given any $R/(x)$-module M, which we can view as an R-module killed by x, then

$$x \cdot \mathrm{Ext}^n_R(M, N) = 0$$

by Lemma 2.7. Thus F^n determines a functor from $R/(x)$-modules to $R/(x)$-modules, as claimed.

To prove 1), consider the short exact sequence of R-modules

$$0 \longrightarrow N \xrightarrow{x} N \longrightarrow N/(x)N \longrightarrow 0.$$

Fix an R-module M killed by x, and apply $\mathrm{Hom}_R(M, -)$ to the short exact sequence above. Note that by Lemma 2.7, x kills $\mathrm{Ext}^i_R(M, N)$ for all i, so we get a long exact sequence

$$0 = \mathrm{Hom}_R(M, N) \longrightarrow \mathrm{Hom}_R(M, N/(x)N) \longrightarrow \mathrm{Ext}^1_R(M, N) \xrightarrow{0} \mathrm{Ext}^1_R(M, N)$$

and thus

$$\mathrm{Hom}_R(M, N/(x)N) \cong \mathrm{Ext}^1_R(M, N) = F^0(M).$$

In fact, this isomorphism is natural thanks to the naturality of the long exact sequence on Ext. Moreover, note that M and $N/(x)N$ are both $R/(x)$-modules, and thus any homomorphism of R-modules $M \longrightarrow N/(x)N$ must be $R/(x)$-linear. Finally, that means we have a natural isomorphism

$$F^0(M) \cong \mathrm{Hom}_{R/(x)}(M, N/(x)N) = \mathrm{Ext}^0_{R/(x)}(M, N/(x)N).$$

To prove 2), let P be any projective module over $R/(x)$. First, we consider the case when P is free over $R/(x)$, say $P = (R/(x))^\alpha$ (with α a cardinal, not necessarily finite). Note that since x is regular over R, then

$$0 \longrightarrow R^\alpha \xrightarrow{x} R^\alpha \longrightarrow 0$$

is a free resolution for P over R, and thus $\mathrm{pdim}_R(P) \leqslant 1$. In particular, for all $n \geqslant 1$ we have

$$F^n(P) = \mathrm{Ext}^{n+1}_R(P, N) = 0.$$

Moreover, if P is any projective module over $R/(x)$, then P is a free summand of some free $R/(x)$-module F by Theorem 4.9 from Homological Algebra. Then since F^n is an additive functor, $F^n(P)$ is a direct summand of $F^n(F) = 0$ for all $n \geqslant 1$, and thus $F^n(P) = 0$.

Finally, we prove 3). Consider a short exact sequence of R-modules killed by x

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0.$$

We get a natural long exact sequence on $\operatorname{Ext}^i_R(-, N)$

$$\begin{array}{ccccccccccc} \operatorname{Hom}_R(A,N) & \to & \operatorname{Ext}^1_R(C,N) & \to & \operatorname{Ext}^1_R(B,N) & \to & \operatorname{Ext}^1_R(A,N) & \to & \operatorname{Ext}^2_R(C,N) & \longrightarrow & \cdots \\ & & \| & & \| & & \| & & \| & & \\ 0 & \longrightarrow & F^0(C) & \longrightarrow & F^0(B) & \longrightarrow & F^0(A) & \longrightarrow & F^1(C) & \longrightarrow & \cdots. \end{array} \quad \square$$

Corollary 6.9. *Let $(R, \mathfrak{m}, k)$ be a noetherian local ring, M a finitely generated R-module, and $x \in \mathfrak{m}$ an element that is both regular on R and on M. Then*

$$\operatorname{injdim}_{R/(x)}(M/(x)M) = \operatorname{injdim}_R(M) - 1.$$

Proof. By Theorem 5.62,

$$\operatorname{injdim}_{R/(x)}(M/(x)M) = \sup\{n \mid \operatorname{Ext}^n_{R/(x)}(k, M/(x)M) \neq 0\}$$

and

$$\operatorname{injdim}_R(M) = \sup\{n \mid \operatorname{Ext}^n_R(k, M) \neq 0\}.$$

The equality follows immediately from Theorem 6.8, as

$$\operatorname{Ext}^n_{R/(x)}(k, M/(x)M) \cong \operatorname{Ext}^{n+1}_{R/(x)}(k, M/(x)M)$$

for all $n \geqslant 0$. $\square$

Corollary 6.10. *Let $(R, \mathfrak{m}, k)$ be a noetherian local ring. Given a regular element $x \in R$, the ring R is Gorenstein if and only if $R/(x)$ is Gorenstein.*

Proof. By Corollary 6.9, $\operatorname{injdim}_R(R)$ is finite if and only if $\operatorname{injdim}_{R/(x)}(R/(x))$ is finite. $\square$

Theorem 6.11. *A noetherian local ring R is Gorenstein if and only if its completion $\widehat{R}$ is Gorenstein.*

Proof. By Theorem 5.65, $\operatorname{injdim}_R(R) = \operatorname{injdim}_{\widehat{R}}(\widehat{R})$, so in particular $\operatorname{injdim}_R(R)$ is finite if and only if $\operatorname{injdim}_{\widehat{R}}(\widehat{R})$ is finite. $\square$

Corollary 6.12. *Every complete intersection ring is Gorenstein.*

Proof. Let R be a complete intersection. By Theorem 6.11, we can reduce to the case when R is complete, so that $R \cong Q/I$ where Q is a regular local ring and I is generated by a regular sequence. Then Q is Gorenstein, by Theorem 6.4, and quotients of Gorenstein rings by regular sequences are Gorenstein by Corollary 6.10, so R is Gorenstein. □

We now have a complete zoo of nice singularities:

Regular rings $\subsetneq$ Complete intersections $\subsetneq$ Gorenstein rings $\subsetneq$ Cohen-Macaulay rings

To see that all these inclusions are strict, we leave the following examples as exercises:

Exercise 66. Give an example of a complete intersection ring that is not regular.

Exercise 67. Let k be any field. Show that

$$k[\![x, y, z]\!]/(x^2 - y^2, x^2 - z^2, xy, xz, yz)$$

is Gorenstein but not a complete intersection.

Exercise 68. Let k be any field. Show that

$$k[\![x, y]\!]/(x^2, xy, y^2)$$

is Cohen-Macaulay but not Gorenstein.

Exercise 69. Give an example of a noetherian local ring that is not Cohen-Macaulay.

Definition 6.13. Let $(R, \mathfrak{m}, k)$ be a Cohen-Macaulay local ring of dimension d. The **type** or **Cohen-Macaulay type** of R is

$$\operatorname{type}(R) = \operatorname{rank}_k \left(\operatorname{Ext}^d_R(k, R) \right).$$

More generally, given a finitely generated module M over a noetherian local ring $(R, \mathfrak{m}, k)$, if $\operatorname{depth}(M) = n$ then the **type** of M is

$$\operatorname{type}(M) = \operatorname{rank}_k \left(\operatorname{Ext}^n_R(k, M) \right).$$

Remark 6.14. Given any Cohen-Macaulay local ring $(R, \mathfrak{m}, k)$ of dimension d, and any finitely generated R-module M,

$$\mu_i(\mathfrak{m}, M) = \operatorname{rank}_k \left(\operatorname{Ext}^i_R(k, M) \right).$$

Thus the type of M is precisely the number of copies of $E(k)$ in (co)homological degree d a minimal injective resolution for M.

Theorem 6.15. *Let $(R, \mathfrak{m}, k)$ be a Cohen-Macaulay local ring. For any regular element $x \in \mathfrak{m}$,*

$$\operatorname{type}(R/(x)) = \operatorname{type}(R).$$

Proof. Note that k is an $R/(x)$-module. Since $\dim(R/(x)) = \dim(R) - 1 = d - 1$,

$$\begin{aligned} \operatorname{type}(R/(x)) &= \operatorname{rank}_k \left(\operatorname{Ext}^{d-1}_{R/(x)}(k, R) \right) \\ &= \operatorname{rank}_k \left(\operatorname{Ext}^{d}_{R}(k, R) \right) \qquad \text{by Theorem 6.8} \\ &= \operatorname{type}(R). \quad \square \end{aligned}$$

Theorem 6.16. *Let $(R, \mathfrak{m}, k)$ be a Cohen-Macaulay local ring. For any system of parameters $\underline{x}$ for R,*

$$\operatorname{type}(R) = \operatorname{rank}_k \left(\operatorname{soc}(R/(\underline{x}) \right).$$

Proof. First, note that $R/(\underline{x})$ is indeed an artinian local ring, since by definition the ideal $(\underline{x})$ is $\mathfrak{m}$-primary. Moreover, since R is Cohen-Macaulay then $\underline{x}$ is a regular sequence, by Theorem 2.35. Applying Theorem 6.15 repeatedly, we conclude that

$$\operatorname{type}(R) = \operatorname{type}(R/(\underline{x})) = \operatorname{rank}_k \left(\operatorname{Hom}_R(k, R/(x)) \right)$$

and by definition of socle, $\operatorname{Hom}_R(k, R/(x)) = \operatorname{soc}(R/(x))$. $\square$

Corollary 6.17. *Let $(R, \mathfrak{m}, k)$ be a Cohen-Macaulay local ring. The ring R is Gorenstein if and only* $\operatorname{type}(R) = 1$.

Proof. Let $\underline{x}$ be a system of parameters for R. Since R is Cohen-Macaulay, $\underline{x}$ is a regular sequence by Theorem 2.35. By Corollary 6.10, R is Gorenstein if and only if $R/(\underline{x})$ is Gorenstein. On the other hand, $R/(\underline{x})$ is an artinian local ring. By Theorem 6.5, $R/(\underline{x})$ is Gorenstein if and only if $\operatorname{soc}(R/(\underline{x}))$ is one-dimensonal over k. Finally, By Theorem 6.16,

$$\operatorname{type}(R) = \operatorname{rank}_k \left(\operatorname{soc}(R/(\underline{x}) \right).$$

We conclude that R is Gorenstein if and only if $\operatorname{type}(R) = 1$. $\square$

Exercise 70. Let $(R, \mathfrak{m}, k)$ be a Gorenstein local ring of dimension d. Show that the minimal injective resolution for R has the form

$$0 \to \bigoplus_{\text{height}(P)=0} E(R/P) \to \bigoplus_{\text{height}(P)=1} E(R/P) \to \cdots \to \bigoplus_{\text{height}(P)=d-1} E(R/P) \to E(R/\mathfrak{m}) \to 0.$$

Equivalently, for all primes P,

$$\mu_i(P, R) = \begin{cases} 1 & \text{if } \operatorname{height}(P) = i \\ 0 & \text{otherwise.} \end{cases}$$

6.2 The canonical module

Theorem 6.18. *Let $(R, \mathfrak{m}, k)$ be a noetherian local ring. If R has a finitely generated MCM module ω of finite injective dimension, then R is Cohen-Macaulay.*

Proof. Since M is finitely generated,

$$\begin{aligned} \operatorname{depth}(R) &= \operatorname{injdim}(\omega) && \text{by Theorem 5.64} \\ &\geqslant \dim(\omega) && \text{by Corollary 5.63} \\ &= \dim(R) && \text{since } \omega \text{ is MCM.} \end{aligned}$$

The inequality $\operatorname{depth}(R) \leqslant \dim(R)$ always holds, so R is Cohen-Macaulay. □

Remark 6.19. If ω is an MCM module of finite injective dimension, the proof of Theorem 6.18 shows that

$$\operatorname{injdim}(\omega) = \dim(R) = \operatorname{depth}(\omega).$$

Definition 6.20. Let $(R, \mathfrak{m}, k)$ be a noetherian local ring of dimension $\dim(R) = d$. A **canonical module** for R is a finitely generated module ω satisfying the following conditions:

- $\operatorname{depth}(\omega) = \dim(R)$, or equivalently, ω is an MCM module.
- $\operatorname{injdim}(\omega) < \infty$.
- $\operatorname{type}(\omega) = 1$, or equivalently, $\operatorname{Ext}^d_R(k, \omega) \cong k$.

Lemma 6.21. *Let $(R, \mathfrak{m}, k)$ be a noetherian local ring of dimension d, and let ω be a finitely generated R-module. Then*

$$\operatorname{Ext}^i_R(k, \omega) = 0 \text{ for all } i \neq d \text{ and } \operatorname{Ext}^d_R(k, \omega) \cong k$$

$$\Updownarrow$$

ω is a canonical module for R.

Proof. Suppose

$$\operatorname{Ext}^i_R(k, \omega) = 0 \text{ for all } i \neq d \text{ and } \operatorname{Ext}^d_R(k, \omega) \cong k.$$

By Theorem 5.62, $\operatorname{injdim}(\omega) = d$ is finite. Moreover, since $\operatorname{Ext}^d_R(k, \omega) \cong k$ then by definition $\operatorname{type}(\omega) = 1$. Finally, by Theorem 2.10, $\operatorname{depth}(\omega) = d$. We conclude that ω is a canonical module for R.

Suppose ω is a canonical module for R. By Remark 6.19, $\operatorname{depth}(\omega) = d$, so by Theorem 2.10

$$\operatorname{Ext}^i_R(k, \omega) = 0 \text{ for all } i < d.$$

By Remark 6.19, $\operatorname{injdim}(\omega) = d$, so

$$\operatorname{Ext}^i_R(k, \omega) = 0 \text{ for all } i > d.$$ □

Remark 6.22. The condition

$$\operatorname{Ext}^i_R(k,\omega) = 0 \text{ for all } i \neq d \text{ and } \operatorname{Ext}^d_R(k,\omega) \cong k$$

is equivalent to

$$\mu_i(\mathfrak{m},\omega) = \begin{cases} 1 & \text{if } i = d \\ 0 & \text{otherwise.} \end{cases}$$

Remark 6.23. Let $(R,\mathfrak{m},k)$ be an artinian local ring. All R-modules have dimension 0, and are thus MCM. Any canonical module for R must be injective, and thus a direct sum of copies of $E = E(k)$. By Exercise 58, E is a finitely generated R-module. Since E^n has type n, we conclude that E is the unique canonical module for R.

Theorem 6.24. *A noetherian local ring $(R,\mathfrak{m},k)$ is Gorenstein if and only if R is a canonical module for R.*

Proof. Let $d = \dim(R)$. If R is Gorenstein, then R is a finitely generated R-module of finite injective dimension. Moreover, by Corollary 6.17 we know that $\operatorname{type}(R) = 1$, meaning that

$$\operatorname{Ext}^d_R(k,R) \cong k.$$

By Exercise 70, $\operatorname{Ext}^i_R(k,R) = 0$ for all $i \neq d$. Thus R is a canonical module for R.

Conversely, if R is a canonical module for R, then R has finite injective dimension, and thus by definition R is Gorenstein. □

Lemma 6.25. *Let $(R,\mathfrak{m},k)$ be a noetherian local ring. If $x \in \mathfrak{m}$ is regular on R and M is an MCM R-module, then x is regular on M.*

Proof. By Theorem 2.36 and the definition of MCM module, every associated prime P of M satisfies

$$\dim(R/P) = \operatorname{depth}(M) = \dim(R).$$

In particular, all the associated primes of M are minimal primes of R, and thus associated primes of R. By Theorem 6.27 from Commutative Algebra I, we conclude that the set of zerodivisors on M is contained in the set of zerodivisors of R. As a consequence, any regular element on R must also be regular on M.

Now given a regular sequence $\underline{x}$ on R, we have shown that x_1 must be regular on M, and since all maximal regular sequences have the same length, we conclude that

$$\operatorname{depth}_{\mathfrak{m}/(x_1)}(M/(x_1)M) = \operatorname{depth}(M) - 1.$$

Since x_1 is regular on R, we also know that

$$\dim(R/(x_1)) = \dim(R) - 1$$

by Exercise 12. Therefore, $M/(x_1)M$ is an MCM module over $R/(x_1)$. Applying an inductive argument now gives us that $\underline{x}$ is a regular sequence on M. □

Theorem 6.26. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring and ω a finitely generated R-module. Let $\underline{x} = x_1, \dots, x_s$ be a sequence that is regular on R and ω. Then ω is a canonical module for R if and only if $\omega/(\underline{x})\omega$ is a canonical module for $R/(\underline{x})$.*

Proof. It suffices to show this holds for $s = 1$ and then apply the statement repeatedly to obtain the general case.

So let x be an element that is regular on both R and ω. By Exercise 12,

$$\dim(R/(x)) = \dim(R) - 1$$

Since x is regular on ω, then

$$\operatorname{depth}(\omega/(x)\omega) = \operatorname{depth}(\omega) - 1.$$

Therefore, ω is MCM over R if and only if $\omega/(x)\omega$ is MCM over $R/(x)$.

By Corollary 6.9,

$$\operatorname{injdim}_{R/(x)}(\omega/(x)\omega) = \operatorname{injdim}_R(\omega) - 1.$$

In particular, $\omega/(x)\omega$ has finite injective dimension over $R/(x)$ if and only if ω has finite injective dimension over R.

Finally, note that x kills k and is regular on both R and M, so by Theorem 6.8

$$\operatorname{Ext}^{n+1}_R(k, \omega) \cong \operatorname{Ext}^n_{R/(x)}(k, \omega/(x)\omega).$$

In particular,

$$\operatorname{rank}_k \left(\operatorname{Ext}^{d-1}_{R/(x)}(k, \omega/(x)\omega) \right) = \operatorname{rank}_k \left(\operatorname{Ext}^d_R(k, \omega) \right).$$

Thus ω is a canonical module for R if and only if $\omega/(x)\omega$ is a canonical module for $R/(x)$. □

By Theorem 6.18, only Cohen-Macaulay rings may have a canonical module. Which Cohen-Macaulay rings have a canonical module? If there is a canonical module for R, is it unique? We start with the question of uniqueness: we will show that over a noetherian local ring, any two canonical modules are isomorphic. First, we need a few lemmas.

Lemma 6.27. *Let M and N finitely generated modules over the a noetherian local ring $(R, \mathfrak{m}, k)$. Let $\underline{x} = x_1, \dots, x_s$ be a regular sequence on N and $\varphi \in \operatorname{Hom}_R(M, N)$. If the induced map $\overline{\varphi} \colon M/(\underline{x})M \longrightarrow N/(\underline{x})N$ is an isomorphism, then φ is an isomorphism.*

Proof. It suffices to do the case $s = 1$, and then apply it repeatedly. So let x be a regular element on N such that $\overline{\varphi}$ is an isomorphism. Since $\operatorname{im}(\varphi) + (x)N = \operatorname{im}(\overline{\varphi}) = N$, $x \in \mathfrak{m}$, and $N \neq 0$ is finitely generated, by NAK φ must be surjective.

If $m \in \ker(\varphi)$, then $m + (x)M \in \ker(\overline{x}) = 0$, so $m \in (x)M$. Let $m = xm_1$ for some $m_1 \in M$. Then since x is regular on N,

$$0 = \varphi(m) = \varphi(xm_1) = x\varphi(m_1) \implies \varphi(m_1) = 0.$$

Repeating the same argument for m_1, and so on, leads to elements $m_n \in \ker(\varphi)$ such that $m = x^n m_n \in (x^n)M$ for all n. By Krull's Intersection Theorem, Theorem 3.37,

$$\bigcap_{n \geqslant 0} (x^n)M = 0$$

and thus φ is injective. □

Theorem 6.28. *Let $(R, \mathfrak{m}, k)$ be a Cohen-Macaulay local ring. Let ω be a canonical module for R, and $M \neq 0$ a finitely generated Cohen-Macaulay module. Set $t = \dim(R) - \dim(M)$. Then*

$$\operatorname{Ext}^i_R(M, \omega) = 0 \quad \text{for all } i \neq t$$

and

$$\operatorname{Ext}^t_R(M, \omega) \neq 0$$

is a Cohen-Macaulay R-module of dimension $\dim(M)$.

Proof. By Ischebeck's Theorem, Theorem 2.23, for all $i < \operatorname{depth}(\omega) - \dim(M) = t$

$$\operatorname{Ext}^i_R(M, \omega) = 0.$$

We will give a proof by induction on $\dim(M)$. If $\dim(M) = 0$, then

$$\operatorname{injdim}(\omega) = \dim(R) = t,$$

so we can compute $\operatorname{Ext}^*_R(-, \omega)$ using an injective resolution of ω of length t, and for all $i \geqslant t + 1$

$$\operatorname{Ext}^i_R(M, \omega) = 0.$$

Since $\dim(M) = 0$ then $\mathfrak{m}$ is the unique minimal prime over $\operatorname{ann}(M)$. By Lemma 2.7,

$$\operatorname{ann}(M) \subseteq \operatorname{ann}\left(\operatorname{Ext}^t_R(M, \omega)\right),$$

and thus $\operatorname{Ext}^t_R(M, \omega)$ has dimension zero. In particular, $\operatorname{Ext}^t_R(M, \omega)$ is Cohen-Macaulay.

Let $d = \dim(R) = t$. To show $\operatorname{Ext}^d_R(M, \omega) \neq 0$, we use induction on $\ell(M)$. If $\ell(M) = 1$, then $M \cong k$, and since $\operatorname{depth}(\omega) = d$, by Theorem 2.10

$$\operatorname{Ext}^d_R(k, \omega) \neq 0.$$

Assume that $\ell(M) \geqslant 2$. Since M has finite length, it has a submodule of length 1, giving us a short exact sequence

$$0 \longrightarrow k \longrightarrow M \longrightarrow N \longrightarrow 0$$

with $\ell(N) = \ell(M) - 1$. Since $\operatorname{injdim}(C) = d$, $\operatorname{Ext}^{d+1}_R(N, C) = 0$. Since $\operatorname{depth}(C) = d$, by Theorem 2.10 we know that $\operatorname{Ext}^d_R(k, C) \neq 0$. Applying $\operatorname{Hom}_R(-, C)$ to the short exact sequence above gives us a long exact sequence

$$\cdots \longrightarrow \operatorname{Ext}^d_R(M, C) \longrightarrow \underbrace{\operatorname{Ext}^d_R(k, C)}_{\neq 0} \longrightarrow \underbrace{\operatorname{Ext}^{d+1}_R(N, C)}_{0} \longrightarrow \cdots.$$

We conclude that $\operatorname{Ext}^d_R(M, C) \neq 0$, finishing the proof in the case when $\dim(M) = 0$.

Now let $\dim(M) \geqslant 1$, and suppose we have proved the statement for all Cohen-Macaulay R-modules of dimension $\dim(M)-1$. We can choose $x \in \mathfrak{m}$ regular on M. Note that x is not in any minimal prime of $\operatorname{ann}(M)$, so $\dim(M/(x)M) = \dim(M)-1$, and moreover

$$\operatorname{depth}(M/(x)M) = \operatorname{depth}(M) - 1.$$

Therefore, $M/(x)M$ is a Cohen-Macaulay module of dimension $\dim(M)-1$. By induction hypothesis,

$$\operatorname{Ext}^i_R(M/(x)M, \omega) = 0 \quad \text{for all } i \neq t.$$

Now consider the short exact sequence

$$0 \longrightarrow M \xrightarrow{x} M \longrightarrow M/(x)M \longrightarrow 0.$$

Applying $\operatorname{Hom}_R(-, \omega)$, we get a long exact sequence

$$\cdots \longrightarrow \operatorname{Ext}^i_R(M,\omega) \xrightarrow{x} \operatorname{Ext}^i_R(M,\omega) \longrightarrow \operatorname{Ext}^{i+1}_R(M/(x)M,\omega) \longrightarrow \cdots.$$

For all $i \neq t$, $\operatorname{Ext}^{i+1}_R(M/(x)M,\omega) = 0$, and thus we get an exact sequence

$$\cdots \longrightarrow \operatorname{Ext}^i_R(M,\omega) \xrightarrow{x} \operatorname{Ext}^i_R(M,\omega) \longrightarrow \underbrace{\operatorname{Ext}^{i+1}_R(M/(x)M,\omega)}_{0} \longrightarrow \cdots.$$

Therefore, $\operatorname{Ext}^i_R(M,\omega) = x \cdot \operatorname{Ext}^i_R(M,\omega)$. Since M and ω are both finitely generated, then so is $\operatorname{Ext}^i_R(M,\omega)$, and by NAK we conclude that $\operatorname{Ext}^i_R(M,\omega) = 0$ for all $i \neq t$.

Now taking $i = t$ and using the induction hypothesis applied to $M/(x)M$, our exact sequence is

$$\underbrace{\operatorname{Ext}^t_R\left(\tfrac{M}{(x)M}, \omega\right)}_{0} \longrightarrow \operatorname{Ext}^t_R(M,\omega) \xrightarrow{x} \operatorname{Ext}^t_R(M,\omega) \longrightarrow \underbrace{\operatorname{Ext}^{t+1}_R\left(\tfrac{M}{(x)M}, \omega\right)}_{\neq 0} \longrightarrow \underbrace{\operatorname{Ext}^{t+1}_R(M,\omega)}_{0}.$$

Therefore, $\operatorname{Ext}^t_R(M,\omega)$ surjects onto a nonzero module, giving us

$$\operatorname{Ext}^t_R(M,\omega) \neq 0.$$

Moreover, x is regular on $A = \operatorname{Ext}^t_R(M,\omega)$, and the exact sequence above is actually

$$0 \longrightarrow A \xrightarrow{x} A \longrightarrow \operatorname{Ext}^{t+1}_R(M/(x)M,\omega) \longrightarrow 0.$$

Thus

$$A/(x)A \cong \operatorname{Ext}^{t+1}_R(M/(x)M,\omega),$$

which by induction hypothesis is Cohen-Macaulay of dimension $\dim(M)-1$. Since x is regular on A, we conclude that A is Cohen-Macaulay of dimension $\dim(M)$. $\square$

Exercise 71. Let M and N be R-modules and x an element that is regular on both R and M. Show that

$$\mathrm{Hom}_R(M, N/(x)N) \cong \mathrm{Hom}_{R/(x)}(M/(x)M, N/(x)N).$$

Theorem 6.29. *Let $(R, \mathfrak{m}, k)$ be a Cohen-Macaulay local ring and let ω and ω' be canonical modules for R. Given a regular sequence $\underline{x}$ on R,*

$$\mathrm{Hom}_R(\omega, \omega') \otimes_R R/(x) \cong \mathrm{Hom}_{R/(x)} \left(\frac{\omega}{(x)\omega}, \frac{\omega'}{(x)\omega'} \right).$$

Proof. First, note that it suffices to show this holds for $s = 1$ and then apply the statement repeatedly to obtain the general case, since

$$-\otimes_R R/(x) \otimes_{R/(x)} \frac{R/(x)}{(x,y)/(x)} \text{ and } -\otimes_R R/(x,y)$$

are naturally isomorphic.

By Theorem 6.28,

$$\mathrm{Ext}^i_R(\omega, \omega') = 0$$

for all $i \neq \dim(R) - \dim(\omega) = 0$.

Let x be a regular element on R. By Lemma 6.25, x is regular on the MCM modules ω and ω'. Thus the short exact sequence

$$0 \longrightarrow \omega' \xrightarrow{x} \omega' \longrightarrow \omega'/(x)\omega' \longrightarrow 0$$

gives rise to the long exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(\omega, \omega') \xrightarrow{x} \mathrm{Hom}_R(\omega, \omega') \longrightarrow \mathrm{Hom}_R(\omega, \omega'/(x)\omega') \longrightarrow \underbrace{\mathrm{Ext}^1_R(\omega, \omega')}_{0}.$$

Therefore,

$$\mathrm{Hom}_R(\omega, \omega'/(x)\omega') \cong \frac{\mathrm{Hom}_R(\omega, \omega')}{x \cdot \mathrm{Hom}_R(\omega, \omega')}.$$

On the other hand, by Exercise 71

$$\mathrm{Hom}_{R/(x)} \left(\frac{\omega}{(x)\omega}, \frac{\omega'}{(x)\omega'}) \right) \cong \mathrm{Hom}_R \left(\omega, \frac{\omega'}{(x)\omega'} \right).$$

Putting this together with the previous isomorphism, we conclude that

$$\begin{aligned} \mathrm{Hom}_{R/(x)}(\omega/(x)\omega, \omega'/(x)\omega') &\cong \mathrm{Hom}_R(\omega, \omega'/(x)\omega') \\ &\cong \frac{\mathrm{Hom}_R(\omega, \omega')}{x \cdot \mathrm{Hom}_R(\omega, \omega')} \\ &\cong \mathrm{Hom}_R(\omega, \omega') \otimes_R R/(x). \quad \square \end{aligned}$$

Theorem 6.30. *Let $(R, \mathfrak{m}, k)$ be a Cohen-Macaulay local ring with canonical modules ω and ω'. Then $\omega \cong \omega'$ and $\mathrm{Hom}_R(\omega, \omega) \cong R$. In fact, the following canonical map is an isomorphism:*

$$\begin{aligned} R &\xrightarrow{\mu} \mathrm{Hom}_R(\omega, \omega) \\ r &\longmapsto \mu_r = (w \mapsto rw). \end{aligned}$$

Proof. Fix a system of parameters $\underline{x}$ for R, which by Theorem 2.35 is a regular sequence on R. By Theorem 6.26, $\frac{\omega}{(x)\omega}$ and $\frac{\omega'}{(x)\omega'}$ are canonical modules for $\overline{R} = R/(\underline{x})$.

On the other hand, $\overline{R}$ is artinian, so by Remark 6.23 there is only one canonical module over $\overline{R}$, $E = E_{\overline{R}}(k)$. In particular,

$$\overline{\omega} := \frac{\omega}{(x)\omega} \cong \frac{\omega'}{(x)\omega'}.$$

Moreover, R is artinian and thus complete. By Matlis duality (more precisely, by Corollary 5.85), we have

$$\mathrm{Hom}_{\overline{R}}(\overline{\omega}, \overline{\omega}) = \mathrm{Hom}_{\overline{R}}(E, E) \cong \overline{R} \cong E.$$

On the other hand, by Theorem 6.29

$$\mathrm{Hom}_R(\omega, \omega') \otimes_R R/(\underline{x}) \cong \mathrm{Hom}_{R/(\underline{x})}\left(\frac{\omega}{(x)\omega}, \frac{\omega'}{(x)\omega'}\right) \cong R/(\underline{x}).$$

By Theorem 3.54 and Theorem 3.55 from Homological Algebra,

$$\mathrm{Hom}_R(\omega, \omega') \otimes_R R/\mathfrak{m} \cong \mathrm{Hom}_R(\omega, \omega') \otimes_R R/(\underline{x}) \otimes_{R/(\underline{x})} R/\mathfrak{m} \cong R/(\underline{x}) \otimes_{R/(\underline{x})} R/\mathfrak{m} \cong R/\mathfrak{m}.$$

By NAK, we conclude that $\mathrm{Hom}_R(\omega, \omega')$ is a cyclic R-module.

Let φ be any generator of $\mathrm{Hom}_R(\omega, \omega')$, and consider the map

$$\begin{aligned} R &\xrightarrow{\psi} \mathrm{Hom}_R(\omega, \omega') \\ 1 &\longmapsto \varphi. \end{aligned}$$

Then $\psi \otimes \overline{R}$ is an isomorphism. By Lemma 6.27, ψ is an isomorphism.

It remains to show that φ itself is an isomorphism, so that $\omega \cong \omega'$. By uniqueness of the canonical module over $\overline{R}$, $\varphi \otimes_R \overline{R}$ is an isomorphism. By Lemma 6.27, φ is an isomorphism.

Finally, consider the canonical map $\mu \colon R \longrightarrow \mathrm{Hom}_R(\omega, \omega)$ sending r to μ_r, multiplication by r. Tensoring with $R/(\underline{x})$ as above gives us the canonical map $\overline{R} \longrightarrow \mathrm{Hom}_{\overline{R}}(E, E)$, which is an isomorphism by Matlis duality, or more precisely by Corollary 5.85. We apply Lemma 6.27 once more and conclude that μ is an isomorphism. □

Thanks to Theorem 6.30, we can now refer to *the* canonical module of R.

Notation 4. Let $(R, \mathfrak{m}, k)$ be a Cohen-Macaulay local ring. If R has a canonical module, we denote it by ω_R.

We now address the question of existence.

Theorem 6.31. *Let S be a Cohen-Macaulay local ring and let I be an ideal such that $R = S/I$ is also Cohen-Macaulay. Let $t = \dim(S) - \dim(R)$. If S has a canonical module ω_S, then R has a canonical module, given by*

$$\omega_R = \operatorname{Ext}_S^t(R, \omega_S).$$

Proof. By Lemma 2.7, $\operatorname{Ext}_S^t(R, \omega_S) \neq 0$ is killed by $I = \operatorname{ann}_S(R)$, and thus an R-module. Moreover, $\operatorname{Ext}_S^t(R, \omega_S)$ is a finitely generated module since R and ω_S are finitely generated S-modules. By Theorem 6.28, $\operatorname{Ext}_S^t(R, \omega_S)$ is a Cohen-Macaulay S-module of dimension $\dim(R)$. Therefore, it is an MCM R-module.

Now consider a minimal injective resolution E of ω_S. By Exercise 62,

$$I = 0 \longrightarrow \frac{\operatorname{Hom}_S(R, E^t)}{\operatorname{im}(\partial_*^{t-1})} \xrightarrow{\partial_*^t} \operatorname{Hom}_S(R, E^{t+1}) \longrightarrow \cdots \longrightarrow \operatorname{Hom}_S(R, E^{\dim(S)}) \longrightarrow 0$$

is an injective resolution for

$$\ker\left(\frac{\operatorname{Hom}_S(R, E^t)}{\operatorname{im}(\partial_*^{t-1})} \xrightarrow{\partial_*^t} \operatorname{Hom}_S(R, E^{t+1})\right) = \frac{\ker(\partial_*^t)}{\operatorname{im}(\partial_*^{t-1})} = \mathrm{H}^t\left(\operatorname{Hom}_S(R, E)\right) = \operatorname{Ext}_S^t(R, \omega_S).$$

In particular, $\operatorname{Ext}_S^t(R, \omega_S)$ has finite injective dimension over R.

To prove $\operatorname{Ext}_S^t(R, \omega_S)$ is a canonical module for R, we still need to check it has type 1. More precisely, we need to calculate

$$\operatorname{Ext}_R^{\dim(R)}(k, \operatorname{Ext}_S^t(R, \omega_S)) = \mathrm{H}^{\dim(R)}\left(\operatorname{Hom}_R(k, I)\right).$$

More precisely, we are looking for the cokernel of

$$\operatorname{Hom}_R(k, \operatorname{Hom}_S(R, E^{\dim(S)-1})) \longrightarrow \operatorname{Hom}_R(k, \operatorname{Hom}_S(R, E^{\dim(S)})) \longrightarrow 0.$$

By Hom–Tensor adjunction, Theorem 3.84 from Homological Algebra, there are natural isomorphisms

$$\operatorname{Hom}_R(k, \operatorname{Hom}_S(R, E^i)) \cong \operatorname{Hom}_S(k \otimes_R R, E^i) \cong \operatorname{Hom}_S(k, E^i).$$

Therefore, we want to compute the cokernel of

$$\operatorname{Hom}_S(k, E^{\dim(S)-1}) \longrightarrow \operatorname{Hom}_S(k, E^{\dim(S)}) \longrightarrow 0.$$

Since E is an injective resolution for ω_S over S, this computes $\operatorname{Ext}_S^{\dim(S)}(k, \omega_S) \cong k$. We conclude that

$$\operatorname{Ext}_R^{\dim(R)}(k, \omega_R) \cong k.$$

□

Remark 6.32. Whenever S is Cohen-Macaulay and $R = S/I$, by Theorem 2.43

$$\dim(S) - \dim(R) = \text{height}(I).$$

Thus Theorem 6.31 says that

$$\omega_R \cong \text{Ext}_S^{\text{height}(I)}(R, \omega_S).$$

Corollary 6.33. *Every complete Cohen-Macaulay local ring R has a canonical module.*

Proof. By Cohen's Structure Theorem, $R \cong Q/I$ for some regular local ring Q. Regular rings are Gorenstein, by Theorem 6.4, so Q has a canonical module $\omega_Q \cong Q$, by Theorem 6.24. By Theorem 6.31, R also has a canonical module. □

Remark 6.34. In fact, by Theorem 6.31 if $R = Q/I$ with Q a regular local ring then the canonical module of R is

$$\omega_R \cong \text{Ext}_Q^{\text{height}(I)}(R, Q).$$

To characterize the rings that have a canonical module, we need the construction known as Nagata's idealization, which according to the introduction to the english edition appears "manywhere" in Nagata's *Local rings* book [Nag62]. We collect here only the facts we need on this topic; for more, see Anderson and Winders' survey [AW09].

Definition 6.35. Let R be a ring and M any R-module. The **Nagata idealization** of (R, M) is the ring $R \rtimes M$ defined as follows:

- As a set, $R \rtimes M = R \times M$.
- The addition operation is given by $(r, m) + (s, n) = (r + s, m + n)$.
- The multiplication operation is given by $(r, m)(s, n) = (rs, sm + rn)$.

Remark 6.36. Note that R is a subring of $R \rtimes M$ via the inclusion $r \mapsto (r, 0)$. Moreover, as an R-module, $R \rtimes M \cong R \oplus M$. On the other hand, the set

$$0 \oplus M = \{(0, m) \in R \rtimes M \mid m \in M\}.$$

is an ideal of $R \rtimes M$, and R is a quotient of $R \rtimes M$:

$$R \rtimes M/0 \oplus M \cong R.$$

Exercise 72. Let $(R, \mathfrak{m})$ be a noetherian local ring and M a finitely generated R-module. Show that $R \rtimes M$ is a noetherian local ring with maximal ideal

$$\mathfrak{m} \rtimes M := \{(r, m) \in R \rtimes M \mid r \in \mathfrak{m}\}.$$

Moreover, show that if R is an artinian, then so is $R \rtimes M$.

In fact, one can show more: the prime ideals of $R \rtimes M$ are of the form $P \rtimes M$ for P a prime ideal in R, and thus $\dim(R \rtimes M) = \dim(R)$.

Exercise 73. Let R be a Cohen-Macaulay local ring with a canonical module ω and consider the Nagata idealization $S = R \rtimes \omega$. Let $\underline{x}$ be a regular sequence on R.

1) Show that the image of $\underline{x}$ in S is a maximal regular sequence in S.

2) Show that
$$S/(\underline{x})S \cong R/(\underline{x}) \rtimes \omega/(\underline{x})\omega.$$

Theorem 6.37. *A Cohen-Macaulay local ring $(R, \mathfrak{m}, k)$ has a canonical module if and only if $R \cong S/I$ for some Gorenstein ring S.*

Proof. ($\Leftarrow$) Suppose $R \cong S/I$ with S Gorenstein. By Theorem 6.24, S has a canonical module, which is S itself. By Theorem 6.31, R has a canonical module, given by $\operatorname{Ext}^{\operatorname{height}(I)}(R, S)$.

($\Rightarrow$) Suppose that R has a canonical module $\omega = \omega_R$. Consider the Nagata idealization $S = R \rtimes \omega$. We will prove that S is a Gorenstein ring. As we noted in Remark 6.36, R is a quotient of S, so this will complete the proof.

Let $\underline{x}$ be a system of parameters on R, which by Theorem 2.35 must be a maximal regular sequence on R. By Exercise 73, the image of $\underline{x}$ in S is a maximal regular sequence on S, and

$$S/(\underline{x})S \cong R/(\underline{x}) \rtimes \omega/(\underline{x})\omega.$$

Moreover, $R/(\underline{x})$ is an artinian ring. By Theorem 6.26, $\omega/(\underline{x})\omega$ is a canonical module for the artinian ring $R/(\underline{x})$, and thus by Remark 6.23 we must have $\omega/(\underline{x})\omega \cong E_{R/(\underline{x})}(k)$.

By Corollary 6.10, S is Gorenstein if and only if $S/(\underline{x})S$ is Gorenstein. By Exercise 72, $S/(\underline{x})S$ is an artinian ring. We have thus reduced the problem that showing that if R is an artinian Gorenstein local ring, and $E = E_R(k)$, then the Cohen-Macaulay local ring

$$S = R \rtimes E$$

has type 1, and is thus a Gorenstein ring.

Consider $(a, x) \in \operatorname{soc}(S)$. By Exercise 72, if $m \in \mathfrak{m}$ then $(m, 0)$ is in the maximal ideal of S, and thus
$$0 = (m, 0)(a, x) = (ma, mx) \implies ma = 0 \text{ and } mx = 0.$$

In particular, $a \in \operatorname{soc}(R)$, and $x \in \operatorname{soc}(E)$.

Suppose $a \neq 0$. By Exercise 59, E is a faithful R-module, meaning that $\operatorname{ann}(E) = 0$. Thus there exists $e \in E$ such that $ae \neq 0$, and

$$(0, e)(a, x) = (0, ae) \neq 0.$$

This contradicts our assumption that $(a, x) \in \operatorname{soc}(S)$, so we must have $a = 0$.

We conclude that all the elements in $\operatorname{soc}(S)$ are of the form $(0, x)$ for some $x \in E$, and

$$\operatorname{soc}(S) \cong \operatorname{soc}(E).$$

Since R is artinian,

$$\begin{aligned} \operatorname{soc}(E) &= \operatorname{Hom}_R(k, E) && \text{by definition of socle} \\ &= E_k(k) && \text{by Lemma 5.45.} \\ &= k && \text{by Corollary 5.11.} \end{aligned}$$

We conclude that $\operatorname{soc}(S) \cong k$. By Corollary 6.17, S is Gorenstein. □

Theorem 6.38. *Let $(R, \mathfrak{m}, k)$ be a Cohen-Macaulay local ring with a canonical module ω_R. For any prime ideal P, R_P has a canonical module, given by*

$$\omega_{R_P} \cong (\omega_R)_P .$$

Proof. By Theorem 6.37, $R = S/I$ for some Gorenstein local ring S. Set

$$h = \dim(S) - \dim(R) = \operatorname{height}(I).$$

By Theorem 6.31,

$$\omega_R \cong \operatorname{Ext}^h_S(R, S).$$

Let $P \in \operatorname{Spec}(R)$, and let Q be the contraction of P in S. First, note that since $R = S/I$ is Cohen-Macaulay, if $Q \in \operatorname{Min}(I)$ then

$$\dim(S/Q) = \dim(S/I).$$

Since S is Cohen-Macaulay, by Theorem 2.43

$$\dim(S) - \operatorname{height}(Q) = \dim(S/Q) = \dim(S/I) = \dim(S) - \dim(I).$$

Therefore, $\operatorname{height}(Q) = \operatorname{height}(I)$ for all minimal primes Q over I. As a consequence, if Q is now any prime containing I, then

$$\operatorname{height}_{S_Q}(IS_Q) = \operatorname{height}_S(I).$$

Using again that S is Cohen-Macaulay, by Theorem 2.43 we have

$$\dim(S_Q) - \dim(R_P) = \operatorname{height}_{S_Q}(IS_Q) = \operatorname{height}_S(I) = \dim(S) - \dim(R) = h.$$

Since S_Q is Gorenstein, S_Q is a canonical module for S_Q. Therefore,

$$(\omega_R)_P \cong (\operatorname{Ext}^h_S(R, S))_P \cong \operatorname{Ext}^h_{S_Q}(R_P, S_Q) \cong \operatorname{Ext}^h_{S_Q}(R_P, S_Q).$$

is a canonical module for R_P, by Theorem 6.31. □

Theorem 6.39. *Let $(R, \mathfrak{m}, k)$ be a noetherian local ring with a canonical module ω_R. Then $\widehat{R}$ has a canonical module, and $\omega_{\widehat{R}} \cong \widehat{\omega_R}$.*

Proof. The completion of a finitely generated R-module is finitely generated, so $\widehat{\omega_R}$ is finitely generated. Since $\dim(R) = \dim(\widehat{R})$ and depth is preserved by completion by Theorem 3.54, $\widehat{\omega_R}$ is an MCM module. By Theorem 5.65, $\widehat{\omega_R}$ has finite injective dimension. In fact, as noted in Remark 5.66, the Bass numbers with respect to $\mathfrak{m}$ are preserved by completion, and thus

$$\operatorname{type}_{\widehat{R}}(\widehat{\omega_R}) = \operatorname{type}_R(\omega_R) = 1.$$

We conclude that $\widehat{\omega_R}$ is a canonical module for $\widehat{R}$. □

Remark 6.40. Alternatively, one can now note that if R has a canonical module, then $R = S/I$ for some Gorenstein ring S. By Theorem 6.11, $\widehat{S}$ is also Gorenstein, and thus $\widehat{R} = \widehat{S}/I\widehat{S}$ has a canonical module. Moreover,

$$\omega_R = \operatorname{Ext}^h_S(R, S)$$

for $h = \operatorname{height}(I)$. Using the same techniques as in Theorem 3.54 and Theorem 5.65, one shows that

$$\widehat{\omega_R} = \operatorname{Ext}^h_S(R, S) \otimes_R \widehat{R} \cong \operatorname{Ext}^h_{\widehat{S}}(\widehat{R}, \widehat{S}).$$

By Theorem 6.31, $\widehat{\omega_R}$ is a canonical module for $\widehat{R}$.

Definition 6.41. A noetherian local ring $(R, \mathfrak{m}, k)$ is **generically Gorenstein** if R_P is Gorenstein for all minimal primes P.

Example 6.42. Recall that a ring R is **reduced** if $\sqrt{(0)} = 0$. All reduced Cohen-Macaulay rings are generically Gorenstein: if $P \in \operatorname{Min}(R)$, then R_P is a field, which is Gorenstein.

Theorem 6.43. *Let $(R, \mathfrak{m}, k)$ be a Cohen-Macaulay local ring with canonical module ω_R. The ring R is generically Gorenstein if and only if canonical module ω_R is isomorphic to an ideal of R.*

Proof. Set $\omega = \omega_R$.

($\Leftarrow$) For each minimal prime P of R, ω_P is a canonical module for R_P by Theorem 6.38. Since R_P is Gorenstein, by Theorem 6.24

$$\omega_P \cong \omega_{R_P} \cong R_P.$$

Since R is Cohen-Macaulay, $\operatorname{Ass}(R) = \operatorname{Min}(R)$ by Theorem 2.36. Consider

$$W = R \setminus \bigcup_{P \in \operatorname{Ass}(R)} P = R \setminus \bigcup_{P \in \operatorname{Min}(R)} P,$$

which consists of all the regular elements on R together with all the units. We have

$$W^{-1}\omega_R \cong \prod \omega_{R_P} \cong \prod R_P \cong W^{-1}R.$$

By Lemma 6.25, all regular elements on R are also regular on ω, so the canonical localization map $\omega \longrightarrow W^{-1}\omega$ is an inclusion. Thus the isomorphism $W^{-1}\omega \cong W^{-1}R$ leads to an inclusion $\alpha : \omega \hookrightarrow W^{-1}R$. Given generators $w_1, \ldots, w_t$ for ω, and $\alpha(w_i) = r_i/s_i$, the map $s_1 \cdots s_t \alpha$ has image in R. Since all the s_i are regular on ω, $s_1 \cdots s_t \alpha$ is an injective map from ω to R. In particular, ω is isomorphic to an ideal of R.

($\Rightarrow$) Suppose that $\omega \cong I$ for some ideal I, and let $P \in \operatorname{Min}(R)$. By Theorem 6.38, ω_P is a canonical module for R_P. Since R_P is artinian, by Remark 6.23

$$I_P \cong \omega_P \cong E_{R_P}(\kappa(P)).$$

On the other hand, $I_P \subseteq R_P$, so

$$\ell(E_{R_P}(\kappa(P))) = \ell(I_P) \leqslant \ell(R_P).$$

By Corollary 5.85, $E_{R_P}(\kappa(P))$ is the Matlis dual of R_P. By Lemma 5.77, the Matlis functor preserves length. Therefore,

$$\ell(E_{R_P}(\kappa(P))) = \ell(I_P) \leqslant \ell(R_P) = \ell(E_{R_P}(\kappa(P))).$$

We conclude that equality must hold throughout, and thus $\ell(I_P) = \ell(R_P)$. Since $I_P \subseteq R_P$, we must have $I_P = R_P$. Therefore, $\omega_P \cong R_P$, so by Theorem 6.24 the ring R_P is Gorenstein. Since this holds for all minimal primes P, we conclude that R is generically Gorenstein. □

Definition 6.44. Let $(R, \mathfrak{m}, k)$ be a Cohen-Macaulay local ring with canonical module ω_R. Any ideal I in R such that $I \cong \omega_R$ is called a **canonical ideal** of R.

Theorem 6.45. *Let $(R, \mathfrak{m}, k)$ be a Cohen-Macaulay local ring with canonical module ω_R and $\dim(R) \geqslant 1$. Suppose that R is generically Gorenstein. If I is a canonical ideal for R, then every associated prime of R/I has height one.*

Proof. Suppose that I is contained in some minimal prime P. Since R_P is Gorenstein,

$$I_P \cong (\omega_R)_P \cong R_P.$$

Let $a \in I_P$ be a generator for the cyclic module I_P. Since

$$\begin{aligned} R_P &\longrightarrow I_P \\ r &\longrightarrow ra \end{aligned}$$

is an isomorphism, $\operatorname{ann}(a) = 0$. On the other hand, every element in $P_P \supseteq I_P$ is a zerodivisor on R_P, and in particular a is a zerodivisor, which contradicts the fact that $\operatorname{ann}(a) = 0$. We conclude that I is not contained in any minimal prime of R, and thus any associated prime of I has height at least 1.

Assume, by contradiction, that $Q \in \operatorname{Ass}(R/I)$ has $\operatorname{height}(Q) \geqslant 2$. On the one hand, R_Q is Cohen-Macaulay of $\dim(R_Q) \geqslant 2$, with canonical module IR_Q, and thus

$$\operatorname{depth}(IR_Q) = \dim(R_Q) \geqslant 2.$$

On the other hand, since Q is associated to I then QR_Q is associated to IR_Q, so

$$\operatorname{depth}(R_Q/IR_Q) = 0.$$

Moreover, R_Q is Cohen-Macaulay, so $\operatorname{depth}(R_Q) = \dim(R_Q) \geqslant 2$. Consider the short exact sequence

$$0 \longrightarrow IR_Q \longrightarrow R_Q \longrightarrow R_Q/IR_Q \longrightarrow 0.$$

By the Depth Lemma, Theorem 2.13,

$$0 = \operatorname{depth}(R_Q/IR_Q) \geqslant \min\{\operatorname{depth}(R_Q), \operatorname{depth}(IR_Q) - 1\}.$$

We conclude that $\operatorname{depth}(IR_Q) \leqslant 1$, which is a contradiction. □

Remark 6.46. Note that the assumption that the existence of a canonical ideal is not interesting when R is artinian: in that case, generically Gorenstein simply means Gorenstein, and thus $\omega_R \cong R$ is indeed an ideal of R, though not a proper one. Thus the assumption that $\dim(R) \geqslant 1$ in Theorem 6.45 is quite mild.

Not all Cohen-Macaulay local rings are quotients of Gorenstein rings; equivalently, not all Cohen-Macaulay rings have a canonical module. However, it is not so easy to write down such an example, since most local rings we typically think of are in fact quotients of regular local rings.

Example 6.47 (Ferrand–Raynaud, [FR70])**.** In their paper *Fibres formelles d'un anneau locale nothérian*, Ferrand and Raynaud give an example of a 1-dimensional local domain $(R, \mathfrak{m}, k)$ such that $\widehat{R}$ is *not* generically Gorenstein. While the construction of their ring is somewhat involved, we will instead focus on the key point: we claim that any such ring is Cohen-Macaulay but does not have a canonical module.

First, we check that R is indeed Cohen-Macaulay: it is a domain, so it contains a regular element, which suffices since $\dim(R) = 1$. Suppose that R has a canonical module ω. Since R is a domain, it is generically Gorenstein, by Example 6.42. By Theorem 6.43, there is an ideal I such that $I \cong \omega$. Since any such I has pure height 1 and R has dimension 1, then I must be $\mathfrak{m}$-primary.

By Theorem 3.61, $\widehat{R}$ is Cohen-Macaulay. By Theorem 6.39, the canonical module of $\widehat{R}$ is $\hat{\omega} \cong \hat{I}$, which is also $\widehat{\mathfrak{m}}$-primary. Thus, $\hat{I}$ is not contained in any $P \in \operatorname{Min}(\widehat{R})$. Therefore, for $P \in \operatorname{Min}(\widehat{R})$, we have

$$\widehat{R}_P = \hat{I}_P \cong \hat{\omega}_P,$$

which contradicts the fact that $\widehat{R}$ is not generically Gorenstein.

When a canonical module does exist, it can be used to prove some very powerful facts.

Exercise 74. Let R be a Cohen-Macaulay ring with a canonical module ω_R. Show that the minimal injective resolution for ω_R has the form

$$0 \to \bigoplus_{\operatorname{height}(P)=0} E(R/P) \to \bigoplus_{\operatorname{height}(P)=1} E(R/P) \to \cdots \to \bigoplus_{\operatorname{height}(P)=d-1} E(R/P) \to E(R/\mathfrak{m}) \to 0.$$

Equivalently, for all primes P,

$$\mu_i(P, \omega_R) = \begin{cases} 1 & \text{if } \operatorname{height}(P) = i \\ 0 & \text{otherwise.} \end{cases}$$

This minimal injective resolution should look very familiar: this is exactly the shape of a minimal injective resolution of a Gorenstein local ring. Thus the canonical module is a sort of substitute for the ring itself when R is not Gorenstein.

The free resolution of the canonical module is also of interest. We leave the following famous results to be proved in the final exam.

Theorem 6.48. *Let Q be a regular local ring and $R = Q/I$ be Cohen-Macaulay with $h = \operatorname{height}(I)$. Let*

$$0 \longrightarrow F_h \xrightarrow{\partial_h} \cdots \longrightarrow F_1 \xrightarrow{\partial_1} F_0 \longrightarrow R \longrightarrow 0$$

be the minimal free resolution for R over Q. Then

$$0 \longrightarrow F_0 \xrightarrow{\partial_0^*} \cdots \longrightarrow F_{h-1} \xrightarrow{\partial_h^*} F_h \longrightarrow \omega_R \longrightarrow 0$$

is a minimal free resolution for ω_R over Q. In particular, ω_R has finite projective dimension, and for all $i \geqslant 0$

$$\beta_i^Q(R) = \beta_{h-i}^Q(\omega_R).$$

It follows that the betti table of a Gorenstein quotient of a regular local ring must be symmetric. In fact, this characterizes Gorenstein rings.

Corollary 6.49. *Let Q be a regular local ring and $R = Q/I$ be Cohen-Macaulay with $h = \operatorname{height}(I)$. The ring R is Gorenstein if and only if for all $i \geqslant 0$*

$$\beta_i^Q(R) = \beta_{h-i}^Q(R).$$

Remark 6.50. A particular useful consequence of Corollary 6.49 is that when $R = Q/I$ is Gorenstein with Q a regular local ring, the last betti number of R over Q must be 1, since

$$\beta_{\operatorname{pdim}_Q(R)}(R) = \beta_0(R) = 1.$$

Exercises

Exercise 75. Let $(R, \mathfrak{m}, k)$ be a Cohen-Macaulay local ring with a canonical module ω_R. Show that $\operatorname{ann}(\omega_R) = 0$.

Lemma 6.51. *Let $(R, \mathfrak{m}, k)$ be a Cohen-Macaulay local ring with a canonical module ω_R. Then*

$$\operatorname{Supp}(\omega_R) = \operatorname{Spec}(R).$$

Appendix A

Macaulay2

There are several computer algebra systems dedicated to algebraic geometry and commutative algebra computations, such as Singular (more popular among algebraic geometers), CoCoA (which is more popular with european commutative algebraists, having originated in Genova, Italy), and Macaulay2. There are many computations you could run on any of these systems (and others), but we will focus on Macaulay2 since it is the most popular computer algebra system among US based commutative algebraists.

Macaulay2, as the name suggests, is a successor of a previous computer algebra system named Macaulay. Macaulay was first developed in 1983 by Dave Bayer and Mike Stillman, and while some still use it today, the system has not been updated since its final release in 2000. In 1993, Daniel Grayson and Mike Stillman released the first version of Macaulay2, and the current stable version if Macaulay2 1.16.

Macaulay2, or M2 for short, is an open-source project, with many contributors writing packages that are then released with the newest Macaulay2 version. Journals like the *Journal of Software for Algebra and Geometry* publish peer-refereed short articles that describe and explain the functionality of new packages, with the package source code being peer reviewed as well.

The National Science Foundation has funded Macaulay2 since 1992. Besides funding the project through direct grants, the NSF has also funded several Macaulay2 workshops — conferences where Macaulay2 package developers gather to work on new packages, and to share updates to the Macaulay2 core code and recent packages.

A.1 Getting started

A Macaulay2 session often starts with defining some ambient ring we will be doing computations over. Common rings such as the rationals and the integers can be defined using the commands `QQ` and `ZZ`; one can easily take quotients or build polynomial rings (in finitely many variables) over these. For example,

```
i1 : R = ZZ/101[x,y]

o1 = R
```

```
o1 : PolynomialRing
```

and

```
i1 : k = ZZ/101;

i2 : R = k[x,y];
```

both store the ring $\mathbb{Z}/101$ as R, with the small difference that in the second example Macaulay2 has named the coefficient field k. One quirk that might make a difference later is that if we use the first option and later set k to be the field $\mathbb{Z}/101$, our ring R is *not* a polynomial ring over k. Also, in the second example we ended each line with a ;, which tells Macaulay2 to run the command but not display the result of the computation — which is in this case was simply an assignment, so the result is not relevant. Lines indicated with as in, where n is some integer, are input lines, whereas lines with an on indicate output lines.

We can now do all sorts of computations over our ring R. We can define ideals in R, and use them to either define a quotient ring S of R or an R-module M, as follows:

```
 i3 : I = ideal(x^2,y^2,x*y)

              2   2
o3 = ideal (x , y , x*y)

o3 : Ideal of R

i4 : M = R^1/I

o4 = cokernel | x2 y2 xy |

                              1
o4 : R-module, quotient of R

i5 : S = R/I

o5 = S

o5 : QuotientRing
```

It is important to note that while R is a ring, R^1 is the R-module R — this is a very important difference for Macaulay2, since these two objects have different types. So S defined above is a ring, while M is a module. Notice that Macaulay2 stored the module M as the cokernel of the map

$$R^3 \xrightarrow{\begin{bmatrix} x^2 & y^2 & xy \end{bmatrix}} R$$

Note also that there is an alternative syntax to write our ideal I from above, as follows:

```
i15 : I = ideal"x2,xy,y2"

              2        2
o15 = ideal (x , x*y, y )

o15 : Ideal of R
```

When you make a new definition in Macaulay2, you might want to pay attention to what ring your new object is defined over. For example, now that we defined this ring S, Macaulay2 has automatically taken S to be our current ambient ring, and any calculation or definition we run next will be considered over S and not R. If you want to return to the original ring R, you must first run the command `use S`.

If you want to work over a finitely generated algebra over one of the basic rings you can define in Macaulay2, and your ring is not a quotient of a polynomial ring, you want to rewrite this algebra as a quotient of a polynomial ring. For example, suppose you want to work over the 2nd Veronese in 2 variables over our field k from before, meaning the algebra $k[x^2, xy, y^2]$. We need 3 algebra generators, which we will call a, b, c, corresponding to x^2, xy, and y^2:

```
 i11 : U = k[a,b,c]

o11 = U

o11 : PolynomialRing

i12 : f = map(R,U,{x^2,x*y,y^2})
                2        2
o12 = map(R,U,{x , x*y, y })

o12 : RingMap R <--- U

i13 : J = ker f

             2
o13 = ideal(b  - a*c)

o13 : Ideal of U

i14 : T = U/J

o14 = T

o14 : QuotientRing
```

Our ring T at the end is isomorphic to the 2nd Veronese of R, which is the ring we wanted.

A.2 Basic commands

Many Macaulay2 commands are easy to guess, and named exactly what you would expect them to be named. If you are not sure how to use a certain command, you can run `viewHelp` followed by the command you want to ask about; this will open an html file with the documentation for the method you asked about. Often, googling "Macaulay2" followed by descriptive words will easily land you on the documentation for whatever you are trying to do.

Here are some basic commands you will likely use:

- `ideal`$(f_1, \dots, f_n)$ will return the ideal generated by $f_1, \dots, f_n$. Here products should be indicated by $*$, and powers with ^. If you'd rather not use ^ (this might be nice if you have lots of powers), you can write `ideal`$(f_1, \dots, f_n)$ instead.
- `map`$(S, R, f_1, \dots, f_n)$ gives a ring map $R \to S$ if R and S are rings, and R is a quotient of $k[x_1, \dots, x_n]$. The resulting ring map will send $x_i \mapsto f_i$. There are many variations of `map` — for example, you can use it to define R-module homomorphisms — but you should carefully input the information in the required format. Try `viewHelp map` in Macaulay2 for more details
- `ker`(f) returns the kernel of the map f.
- `I + J` and `I*J` return the sum and product of the ideals I and J, respectively.
- `A = matrix`$\{\{a_{1,1}, \dots, a_{1,n}\}, \dots, \{a_{m,1}, \dots, a_{m,n}\}\}$ returns the matrix

$$A = \begin{pmatrix} a_{1,1} & \dots & a_{1,n} \\ & \ddots & \\ a_{m,1} & \dots & a_{m,n} \end{pmatrix}$$

A.3 Complexes in Macaulay2

There are two different ways to do computations involving complexes in Macaulay2: using `ChainComplexes`, or the new (and still incomplete) `Complexes` package. To use `Complexes`, you must first load the `Complexes` package, while the `ChainComplexes` methods are automatically loaded with Macaulay2.

A.3.1 Chain Complexes

To create a new chain complex by hand, we start by setting up R-module maps.

```
i1 : R = QQ[a,b];

i2 : d1 = map(R^1, R^2, {{a,b}})
```

```
o2 = | a b |
             1       2
o2 : Matrix R  <--- R

i3 : d2 = map(R^2, R^1, {{-b},{a}})

o3 = | -b |
     | a  |
             2       1
o3 : Matrix R  <--- R
```

Keep in mind that the syntax of map is a bit funny: we write `map(target,source,matrix)`. To make sure we set up the next map in a way that is composable with d_1, we can use the methods `source` and `target`:

```
i3 : d1 = map(source d0, R^1, {{-b},{a}})

o3 = | -b |
     | a  |
             2       1
o3 : Matrix R  <--- R
```

We can also double check our maps do indeed map a complex, by checking the composition $d_1 \circ d_2$:

```
i4 : d1 * d2 == 0

o4 = true
```

So now we are ready to set up our new chain complex.

```
i5 : C = new ChainComplex

o5 = 0

o5 : ChainComplex

i6 : C#0 = target d1

      1
o6 = R

o6 : R-module, free

i7 : C#1 = target d2

      2
```

```
o7 = R

o7 : R-module, free

i8 : C#2 = source d2

      1
o8 = R

o8 : R-module, free
```

Given a chain complex `C`, we can ask Macaulay2 what our complex is by simply running the name of the complex:

```
i9 : C

      1      2      1
o9 = R  <-- R  <-- R

     0      1      2

o9 : ChainComplex
```

Or we can ask for a better visual description of the maps, using `C.dd`:

```
i10 : C.dd

           1         2
o10 = 0 : R  <----- R  : 1
                0

           2         1
      1 : R  <----- R  : 2
                0

o10 : ChainComplexMap
```

We can also set up the same complex in a more compact way, by simply feeding the maps we want in order. Macaulay2 will automatically place the first map with the target in homological degree 0 and the source in degree 1.

```
11 : D = chainComplex(d1,d2)

       1      2      1
o11 = R  <-- R  <-- R

      0      1      2
```

```
o11 : ChainComplex
```

Notice this is indeed the same complex.

```
i12 : D.dd

           1               2
o12 = 0 : R  <----------- R  : 1
                | a b |

           2              1
      1 : R  <---------- R  : 2
                | -b |
                | a  |

o12 : ChainComplexMap
```

We can also ask Macaulay2 to compute the homology of our complex:

```
i13 : HH D

o13 = 0 : cokernel | a b |

      1 : subquotient (| b  |, | -b |)
                       | -a |  | a  |
      2 : image 0

o13 : GradedModule
```

Or we could simply ask for the homology in a specific degree:

```
i14 : HH_0 D

o14 = cokernel | a b |
                                1
o14 : R-module, quotient of R
```

A.3.2 The Complexes package

To use this functionality, you must first load the `Complexes` package.

```
i15 : needsPackage "Complexes";

o15 = Complexes

o15 : Package
```

We can use our maps from above to set up a complex with the same maps. We feed a list of the maps we want to use to the method `complex`.

```
i16 : F = complex({d1,d2})

       1      2      1
o16 = R  <-- R  <-- R

      0      1      2

o16 : Complex
```

We cam read off the maps and the homology in our complex using the same commands as we use with `chainComplexes`, although the information returned gets presented in a slightly different fashion.

```
i17 : HH F

o17 = cokernel | a b | <-- subquotient (| b  |, | -b |) <-- image 0
                                        | -a |  | a  |
      0                                                        2
                             1

o17 : Complex

i18 : F.dd

           1               2
o18 = 0 : R  <----------- R  : 1
                | a b |

           2              1
      1 : R  <---------- R  : 2
                | -b |
                | a  |

o18 : ComplexMap
```

If we want to set up our complex starting in a different homological degree, we can do the following:

```
i19 : G = complex({d1,d2}, Base => 7)

       1      2      1
o19 = R  <-- R  <-- R
```

```
      7      8      9

o19 : Complex

i20 : H = complex({d1,d2}, Base => -13)

       1       2       1
o20 = R   <-- R   <-- R

      -13     -12     -11

o20 : Complex
```

A.3.3 Maps of complexes

Suppose we are given two complexes `C` and `D` and a map of complexes `f : C` $\longrightarrow$ `D`. The routine `map` can be used to define f using `chainComplexes`: it receives the target `D`, the source `D`, and a function `f` that returns f_i when we compute `f(i)`.

```
i1 : R = QQ[a,b];

i2 : c1 = map(R^0,R^1,0);

                     1
o2 : Matrix 0 <--- R

i3 : c2 = map(R^1, R^2, {{a,b}});

              1       2
o3 : Matrix R  <--- R

i4 : c3 = map(R^2, R^1, {{-b},{a}});

              2       1
o4 : Matrix R  <--- R

i5 : c4 = map(R^1, R^0, 0);

              1
o5 : Matrix R  <--- 0

i6 : C = chainComplex(c1,c2,c3,c4);

i7 :
     d1 = map(R^0,R^1,0);
```

```
                    1
o7 : Matrix 0 <--- R

i8 : d2 = id_(R^1);

             1       1
o8 : Matrix R  <--- R

i9 : d3 = map(R^1, R^0, 0);

             1
o9 : Matrix R  <--- 0

i10 : d4 = map(R^0, R^0, 0);

o10 : Matrix 0 <--- 0

i11 : D = chainComplex(d1,d2,d3,d4)

             1      1
o11 = 0 <-- R  <-- R  <-- 0 <-- 0

      0     1      2      3     4

o11 : ChainComplex

i12 :
      f0 = map(R^0, R^0, 0);

o12 : Matrix 0 <--- 0

i13 : f1 = map(R^1, R^1, matrix{{0_R}});

              1       1
o13 : Matrix R  <--- R

i14 : f2 = map(R^2, R^1, {{b},{-a}});

              2       1
o14 : Matrix R  <--- R

i15 : f3 = map(R^1, R^0, 0);

              1
```

```
o15 : Matrix R  <--- 0

i16 : f4 = map(R^0, R^0, 0);

o16 : Matrix 0 <--- 0

i17 : f = map(C,D,i -> if i==0 then f0 else(
      if i==1 then f1 else (
      if i==2 then f2 else (
      if i == 3 then f3 else (
      if i==4 then f4)))))

o17 = 0 : 0 <----- 0 : 0
              0

           1         1
      1 : R  <----- R  : 1
                0

           2              1
      2 : R  <---------- R  : 2
                | b  |
                | -a |
           1
      3 : R  <----- 0 : 3
                0

      4 : 0 <----- 0 : 4
              0

o17 : ChainComplexMap
```

Here's what we can do if we prefer to write a list with the maps in `f`:

```
i18 : f = map(C,D,i -> {f0,f1,f2,f3,f4}_i)

o18 = 0 : 0 <----- 0 : 0
              0

           1         1
      1 : R  <----- R  : 1
                0

           2              1
      2 : R  <---------- R  : 2
```

```
               | b  |
               | -a |

            1
      3 : R  <----- 0 : 3
               0

      4 : 0 <----- 0 : 4
             0

o18 : ChainComplexMap
```

If we prefer to do the same with the `Complexes` package, one advantage is that `map` *does* receive `(target, source, list of maps)`.

```
i42 : C = complex({c1,c2,c3,c4});

i43 : D = complex({d1,d2,d3,d4});

i44 : f = map(C,D,{f0,f1,f2,f3,f4})

              2                 1
o44 = 2 : R  <---------- R  : 2
               | b  |
               | -a |

o44 : ComplexMap
```

Index

Nomenclature

$\beta_i(M)$ ith betti number of R

$\beta ij(M)$ graded betti numbers of M

$\operatorname{depth}(M)$ depth of M (wrt the maximal ideal)

$\operatorname{depth}_I(M)$ I-depth of M

$\operatorname{gldim}(R)$ global dimension of R

$\mathrm{H}_i(\underline{x}; M)$ ith koszul homology of M wrt $\underline{x}$

$\operatorname{injdim}(M)$ injective dimension of M

$\kappa(P)$ R_P/PR_P, the residue field of R_P

$\operatorname{kos}(x_1, \ldots, x_n)$ the koszul complex on $x_1, \ldots, x_n$

$\mu_i(P, M)$ ith bass number of M wrt P

ω_R canonical module for R

$\operatorname{cx}(M)$ complexity of M

$\operatorname{rank} M$ rank of the module M

$\operatorname{Supp}(M)$ support of M

$\underline{x}$ shorthand for $x_1, \ldots, x_n$

$\widehat{R}$ completion of R (wrt the maximal ideal)

$\widehat{R}^I$ I-adic completion of R

$E(M)$ injective hull of M

K^R koszul complex on a minimal generating set for $\mathfrak{m}_R$

$M^\vee$ $\operatorname{Hom}_R(M, E(k))$, the Matlis dual of M

RLR regular local ring

SOP system of parameters

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