Problem Set 4

Due Wednesday, April 22, 2026

Instructions: You are encouraged to work together on these problems, but each student should hand in their own final draft, written in a way that indicates their individual understanding of the solutions. Never submit something for grading that you do not completely understand. You cannot use any resources besides me, your classmates, and our course notes.

I will post the .tex code for these problems for you to use if you wish to type your homework. If you prefer not to type, please *write neatly*. As a matter of good proof writing style, please use complete sentences and correct grammar. You may use any result stated or proven in class or in a homework problem, provided you reference it appropriately by either stating the result or stating its name (e.g. the definition of depth or Hilbert's Syzygy Theorem). Please do not refer to theorems by their number in the course notes, as that can change.

Turn in **4 problems** of your choosing. Any problem you do not turn in is now a known theorem.

Problem 1. Let $(R, \mathfrak{m})$ be a noetherian local ring. Show that if exists a nonzero finitely generated injective R-module E, then $\dim(R) = 0$.

Problem 2. Let k be a field and $R = k[\![x]\!]$. Find the injective dimension and Bass numbers of R, and write down the modules in a minimal injective resolution for R.

Problem 3. Let $(R, \mathfrak{m}, k)$ be a noetherian local ring. Show that if R is injective then $R \cong E(R/\mathfrak{m})$.

Problem 4. Let $(R, \mathfrak{m}, k)$ be a noetherian local ring.

a) Show that $\operatorname{ann}(E(k)) = 0$.

b) Give an example of such a ring R and a nonzero injective R-module I such that $\operatorname{ann}(I) \neq 0$.

Problem 5. Let k be a field.

a) Give an example (with proof) of a complete intersection ring A that is not regular.

b) Show that
$$B = k[\![x, y, z]\!]/(x^2 - y^2, x^2 - z^2, xy, xz, yz)$$
is Gorenstein but not a complete intersection.

c) Let k be any field. Show that
$$C = k[\![x, y]\!]/(x^2, xy, y^2)$$
is Cohen-Macaulay but not Gorenstein.

d) Give an example (with proof) of a noetherian local ring B that is not Cohen-Macaulay.