Dimension and height

The main goal of this worksheet is to provide a hands on refresher on dimension and height.

Hands-on examples

(1) Let k be a field. Find, with proof, the heights of the following ideals.
   (a) The ideal (xy, yz, zw) in $k[\![x, y, z, w]\!]$.
   (b) The ideal (x^2, xy) in $k[\![x, y, z]\!]$.
   (c) The ideal (xy, xz) in $k[x, y, z]$.
   (d) The kernel of the ring homomorphism $f\colon k[a, b, c, d] \longrightarrow k[s, t]$ given by

$$f(a) = s^3 \quad f(b) = s^2 t \quad f(c) = st^2 \quad f(d) = t^3.$$

(2) For each of the following rings, find all the possible heights of maximal ideals:
   (a) The maximal ideals of $\mathbb{Z}[\sqrt{-13}]$.
   (b) The maximal ideals of $k[x, y, z]/(xy, xz)$.

(3) Let k be any field. Find, with proof, the dimension of each of the following rings:
   (a) $A = \mathbb{Z}[x]_{(13)}$.
   (b) $B = \mathbb{Z}[x]_{(13,x)}$.
   (c) $C = k[x, y, z]/(xy, yz)$.
   (d) $D = k[x, y, z, w]/(x^2, xy, yz, zw, w^2)$.
   (e) $E = k[x^2u, xyu, y^2u, x^2v, xyv, y^2v] \subseteq k[x, y, u, v]$.
   (f) $F = \mathbb{Z}[\sqrt{-17}]$.
   (g) $G = k[s^2, st, t^2] \subseteq k[s, t]$.
   (h) $H = k[s^3, s^2t, st^2, t^3] \subseteq k[s, t]$.
   (i) $I = k[s^3, s^2t, t^3] \subseteq k[s, t]$.