Worksheet 1: Free resolutions

Facts about free resolutions

The facts below are theorems you can use in our class whether you have proved them or not.

(1) Let R be a local/graded ring. Show that for all finitely generated (graded) R-modules M and all $i \geqslant 1$, if F if the minimal free resolution for M, then there is a natural short exact sequence

$$0 \longrightarrow \Omega_{i+1}(M) \longrightarrow F_i \longrightarrow \Omega_i(M) \longrightarrow 0.$$

(2) Let R be a local/graded domain and M a finitely generated (graded) R-module. Show that

$$\beta_i(M) = \operatorname{rank}\left(\Omega_i(M)\right) + \operatorname{rank}\left(\Omega_{i+1}(M)\right).$$

(3) Let I be any ideal in a local ring R. How do the minimal free resolutions of I and R/I relate to each other? How about $\operatorname{pdim}(I)$ and $\operatorname{pdim}(R/I)$?

(4) From Problem Set 1: Let R be a local domain and I an ideal with $\operatorname{pdim}(R/I) < \infty$. Show that

$$\sum_{i \geqslant 0} \beta_{2i}(R/I) = \sum_{i \geqslant 0} \beta_{2i+1}(R/I).$$

Computing free resolutions

The goal of this section is to compute some explicit free resolutions by hand. You are encouraged to use facts from above, whether you have proved them or not.

(5) Let k be a field and let f be any nonunit homogeneous element in $R = k[x_1, \dots, x_d]$. Find the minimal free resolution of $R/(f)$.

(6) Let k be a field and let f, g be any nonunit homogeneous elements in $R = k[x_1, \dots, x_d]$ that are not multiples of each other. Find the minimal free resolution of $R/(f, g)$.

(7) Let $R = k[x, y, z]$ and $M = R/(xy, xz, yz)$. We will later show that $\operatorname{pdim}(M) = 2$, which you can for now take for granted. Find the minimal free resolution of M, with proof.