Worksheet 2: Depth and Cohen-Macaulay rings

Regular sequences and depth

(1) Let k be a field. For each of the following local rings R, find an explicit maximal regular sequence on R:
 (a) $R = k[\![x]\!]/(x^2)$.
 (b) $R = k[\![x, y]\!]/(xy)$.
 (c) $R = k[\![x, y]\!]/(x^2, xy)$.
 (d) $R = k[\![t^2, t^3]\!] \subseteq k[\![t]\!]$.

Cohen-Macaulay rings

A noetherian local ring $(R, \mathfrak{m})$ is **Cohen-Macaulay** if $\operatorname{depth}(R) = \dim(R)$.

(2) Let k be a field. Determine whether each of following local rings is Cohen-Macaulay:
 (a) $R = k[\![x]\!]/(x^2)$.
 (b) $R = k[\![x, y]\!]/(xy)$.
 (c) $R = k[\![x, y]\!]/(x^2, xy)$.
 (d) $R = k[\![t^2, t^3]\!] \subseteq k[\![t]\!]$.

(3) Let $R = k[\![x, y]\!]/(x^2, xy)$ and $M = R/(x)$. Find $\operatorname{depth}(M)$ and $\dim(M)$, and compare them to what you found about R.

Grade

(4) For each of the following ideals I, compute $\operatorname{grade}(I)$ by finding an explicit maximal regular sequence inside I:
 (a) $I = (xy, xz)$ in $R = k[\![x, y]\!]$.
 (b) $I = (xy, xz, yz)$ in $R = k[\![x, y, z]\!]$.

Auslander–Buchsbaum formula

Theorem 1 (The Auslander–Buchsbaum formula)**.** *Let $(R, \mathfrak{m}, k)$ be a notherian local ring and let $M \neq 0$ be a finitely generated R-module with $\operatorname{pdim}(M) < \infty$. Then*

$$\operatorname{depth}(M) + \operatorname{pdim}(M) = \operatorname{depth}(R).$$

(5) Let $R = k[x, y, z]$ and $I = (xy, xz, yz)$. Find $\operatorname{pdim}(R/I)$.

(6) Prove the Auslander–Buchsbaum formula holds when $\operatorname{depth}(R) = 0$.

(7) Let $R = k[x_1, \dots, x_d]$ with k a field and consider nonzero $f, g \in R$. What are the possible values for $\operatorname{depth}(R/(f, g))$?

More about depth

(8) Show that if $\operatorname{depth}_I(M) = 0$, then $\operatorname{Ext}^0_R(R/I, M) \neq 0$.

(9) Show that for all R-modules M, $\operatorname{ann}(M) \subseteq \operatorname{Ext}^i(M, N)$.

(10) Prove that $\operatorname{depth}_I(M) = \min\{i \mid \operatorname{Ext}^i_R(R/I, M) \neq 0\}$.

Ingredients for the Auslander–Buchsbaum formula

(11) Given a noetherian local ring $(R, \mathfrak{m}, k)$ and a finitely generated R-module $M \neq 0$ of finite projective dimension, show that if $x \in R$ is regular on both R and M, then $M/(x)M$ has finite projective dimension over $R/(x)$, given by

$$\operatorname{pdim}_{R/(x)}(M/(x)) = \operatorname{pdim}_R(M).$$