Worksheet 3: Identifying Cohen-Macaulay rings

Cohen-Macaulay rings

(1) Let k be a field. Determine which of following rings is regular or Cohen-Macaulay:

(a) $k[\![x,y,z]\!]$
(b) $k[\![x]\!]/(x^2)$
(c) $k[\![x,y]\!]/(xy)$
(d) $k[\![x,y]\!]/(x^2,xy)$
(e) $k[\![x,y,z]\!]/(x^3,y^3,z^3)$
(f) $k[\![x,y,z]\!]/(x^3+y^3+z^3)$
(g) $k[\![x,y,z]\!]/(xy^3,y^4,y^3z)$
(h) $k[x,y,z,a,b]/(x,y,z)\cap(a,b)$
(i) $k[x,y,z,a,b]/(x^2,y^3,z^4)\cap(a^5,b^6)$
(j) $\mathbb{Z}[x,y]/(x^2-y^2)$
(k) $\mathbb{Z}[x]/(x^2-1)$

Other problems

(2) Let $R=k[\![x,y]\!]/(x^2,xy)$ and $M=R/(x)$. Find $\operatorname{depth}(M)$ and $\dim(M)$. What do these say about M?

(3) For each of the following ideals I, compute $\operatorname{grade}(I)$ by finding an explicit maximal regular sequence inside I:
(a) $I=(xy,xz)$ in $R=k[\![x,y]\!]$.
(b) $I=(xy,xz,yz)$ in $R=k[\![x,y,z]\!]$.

(4) Let $R=k[x_1,\dots,x_d]$ with k a field and consider nonzero $f,g\in R$. What are the possible values for $\operatorname{depth}(R/(f,g))$?