Final Exam

Due Friday, May 1, 2026

Instructions: Write complete and clear solutions to each problem. If you use Macaulay2, email me a .m2 file with your work, containing comments. The only allowed resources are me, Macaulay2, and our class notes. You can work on the problems with your classmates, but only after spending at least min{time it takes you to solve the problem, 1 hour} per problem.

Turn in **5 problems** of your choosing. Any problem you do not turn is now a known theorem.

Problem 1. Let $(R, \mathfrak{m}, k)$ be a Gorenstein local ring. Show that the minimal injective resolution for R has the form

$$0 \longrightarrow \bigoplus_{\text{height}(P)=0} E(R/P) \longrightarrow \bigoplus_{\text{height}(P)=1} E(R/P) \longrightarrow \cdots \longrightarrow \bigoplus_{\text{height}(P)=\dim(R)-1} E(R/P) \longrightarrow E(R/\mathfrak{m}) \longrightarrow 0.$$

Problem 2. Let $(R, \mathfrak{m}, k)$ be a Cohen-Macaulay ring with a canonical module ω_R. Show that R is Gorenstein if and only if ω_R is a cyclic R-module.

Problem 3. Let $(Q, \mathfrak{m}, k)$ be a regular local ring, and $R = Q/I$ Cohen-Macaulay with $\text{pdim}_Q(R) = n$. Show that R is Gorenstein if and only if the betti sequence for R over Q is symmetric, meaning that for all i

$$\beta_i^Q(R) = \beta_{n-i}^Q(R).$$

Hint: Apply $\text{Hom}_Q(-, Q)$ to the minimal free resolution for R over Q.

Problem 4. Let $(Q, \mathfrak{m}, k)$ be a regular local ring of dimension d and let $R = Q/I$ with $I \subseteq \mathfrak{m}^2$.

a) Show that if R is a Cohen-Macaulay ring of dimension $d-1$, then R is a hypersurface.

b) Show that if R is a Gorenstein ring of dimension $d-2$, then R is a complete intersection.

Problem 5. Consider the nonstandard graded ring

$$R = \mathbb{Q}[t^{13}, t^{42}, t^{73}].$$

Write an explanation of your calculations in the .pdf, and include an auxiliary .m2.

a) Use Macaulay2 to find a minimal presentation for R of the form $R = \mathbb{Q}[x, y, z]/I$, where I is a homogeneous ideal in $\mathbb{Q}[x, y, z]$ under an appropriate choice of grading.

b) Can I be generated by a regular sequence?

c) Is R Gorenstein?

d) Is R Cohen-Macaulay? Explain your answer with a proof, and confirm it with Macaulay2.

e) Let $\mathfrak{m} = (x, y, z)$. Does $R_{\mathfrak{m}}$ have a canonical module? If so, find a presentation for $\omega_{R_{\mathfrak{m}}}$.

Problem 6. Let R be a noetherian ring of dimension 1.

a) Show that if R is reduced,[1] then R is Cohen-Macaulay.

b) Show that if R is local, then R has a finitely generated MCM module.

[1] Recall that a ring is reduced if $\sqrt{(0)} = (0)$.