

## Problem Set 1

Turn in any **4** of the following problems. Slightly more challenging problems are indicated by  $(\star)$ . You are encouraged to work together on these problems, but each student should hand in their own final draft, written in a way that indicates their individual understanding of the solutions. Never submit something for grading that you do not completely understand. You cannot use any resources besides me, your classmates, and our course notes.

**Problem 1.** Consider the category  $R\text{-mod}$ .

- a) Show that a homomorphism of  $R$ -modules is injective if and only if it is a mono in  $R\text{-mod}$ , and surjective if and only if it is an epi in  $R\text{-mod}$ .
- b) Show that the homomorphism of  $\mathbb{Z}$ -modules  $\mathbb{Z} \xrightarrow{2} \mathbb{Z}$  is monic but has no left inverse in  $\mathbb{Z}\text{-mod}$ .
- c) Show that the canonical homomorphism  $\mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z}$  is epic but has no right inverse in  $\mathbb{Z}\text{-mod}$ .

**Problem 2.**

- a) Show that in any category, every isomorphism is both an epi and a mono.
- b) Show that the usual inclusion  $\mathbb{Z} \hookrightarrow \mathbb{Q}$  is an epi in the category **Ring**.  
This *should* feel weird: it says being epi and being surjective are *not* the same thing.
- c) Show that the canonical projection  $\mathbb{Q} \rightarrow \mathbb{Q}/\mathbb{Z}$  is a mono in the category of divisible abelian groups.<sup>1</sup>  
Again, this is very strange: it says being monic and being injective are *not* the same thing.

**Problem 3.** Suppose that  $\mathcal{C}$  and  $\mathcal{D}$  are concrete categories and  $F: \mathcal{C} \rightarrow \mathcal{D}$  is a covariant functor.

- a) Show that if  $\alpha$  is an arrow in  $\mathcal{C}$  that has a left inverse, then  $F(\alpha)$  is an injective function.
- b) Show that if  $\alpha$  is an arrow in  $\mathcal{C}$  that has a right inverse, then  $F(\alpha)$  is a surjective function.
- c) Use part (a) to show<sup>2</sup> that there is no covariant functor  $F: \mathbf{Grp} \rightarrow \mathbf{Grp}$  that, on objects, maps a group to its center.

**Problem 4.** We will show that every functor creates isos, and fully faithful functors reflect isos. Let  $F: \mathcal{C} \rightarrow \mathcal{D}$  be a functor.

- a) Show that if  $f$  is an iso in  $\mathcal{C}$ , then  $F(f)$  is an iso in  $\mathcal{D}$ .
- b) Show that if  $X$  and  $Y$  are isomorphic objects in  $\mathcal{C}$ , then  $F(X)$  and  $F(Y)$  are isomorphic in  $\mathcal{D}$ .
- c) Suppose  $F$  is fully faithful. Show that if  $F(f)$  is an iso, then  $f$  is an iso.
- d) Let  $F$  be fully faithful. Show that if  $F(X)$  and  $F(Y)$  are isomorphic in  $\mathcal{D}$ , then  $X$  and  $Y$  are isomorphic in  $\mathcal{C}$ .
- e) Find an example to show that a faithful functor need not reflect isomorphisms.

<sup>1</sup>An abelian group  $A$  is divisible if for every  $a \in A$  and every positive integer  $n$  there exists  $b \in A$  such that  $nb = a$ .

<sup>2</sup>Hint: You might consider some appropriate inclusion of the group  $\mathbb{Z}/2$  into the symmetric group  $\mathbb{S}_3$ .

**Problem 5.** Let  $\mathbf{Set}^\infty$  be the full subcategory of  $\mathbf{Set}$  consisting of all infinite sets. Let

$$F: \mathbf{Set}^\infty \rightarrow \mathbf{Set}^\infty$$

be the functor that on objects is given by the rule  $F(S) = S \times S$ , and on morphisms is given by  $F(f) = (f, f)$ . Show that there is no natural isomorphism  $\eta: F \Rightarrow 1_{\mathbf{Set}^\infty}$ .

Note: whenever  $S$  is an infinite set, there is a bijection between  $S \times S$  and  $S$ ; that is not the issue.

**Problem 6.** Given  $X$  a partially ordered set, let  $\mathbf{PO}(X)$  denote the corresponding category.

- a) Let  $Y$  be a set, and  $\mathbb{P}(Y)$  be its power set, which we can view as a partially ordered set under  $\subseteq$ . For any two subsets  $A$  and  $B$  of  $Y$ , show that their product in  $\mathbf{PO}(\mathbb{P}(Y))$  is the set  $A \cap B$ , and that their coproduct is  $A \cup B$ .
- b) Generalize the previous example: show in  $\mathbf{PO}(X)$ , coproducts agree with least upper bounds and products agree with greatest lower bounds.

**Problem 7.** Let  $R$  be a ring and let  $M$  be an  $R$ -module.

- a) Let

$$M_1 \supseteq M_2 \supseteq M_3 \supseteq \cdots$$

be a descending chain of submodules of  $M$ , which we can think of as an inverse system in the natural way. Show that the limit of this inverse system is the intersection of the submodules.

- b) Let

$$M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots$$

be an ascending chain of submodules of  $M$ , which we can think of as a direct system in the natural way. Show that the colimit of this direct system is the union of the submodules.

**Problem 8.**

- a) Interpret the notion of pullback as a limit and a pushout as a colimit. More precisely, describe a partially ordered set and corresponding inverse system or direct system whose limit or colimit is the same as a pushout or pullback.
- b) Explicitly describe pullbacks and pushouts in  $R\text{-mod}$ .