Problem Set 2

Turn in any **4** of the following problems. Slightly more challenging problems are indicated by $(\star)$. You are encouraged to work together on these problems, but each student should hand in their own final draft, written in a way that indicates their individual understanding of the solutions. Never submit something for grading that you do not completely understand. You cannot use any resources besides me, your classmates, and our course notes.

Problem 1 (The Five Lemma)**.** Consider the following commutative diagram of R-modules with exact rows:

$$\begin{array}{ccccccccc}
A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\
{\scriptstyle a}\downarrow & & {\scriptstyle b}\downarrow & & {\scriptstyle c}\downarrow & & {\scriptstyle d}\downarrow & & {\scriptstyle e}\downarrow \\
A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E'
\end{array}$$

Show that if a, b, d, and e are isomorphisms, then c is an isomorphism.

Problem 2.

a) Let π denote the canonical projection map from $\mathbb{Z}$ to $\mathbb{Z}/2\mathbb{Z}$. Show that the chain map $f\colon F \longrightarrow G$ given by

$$\begin{array}{rccccccccccc}
F = & \cdots & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{2} & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \pi} & & \downarrow & & \\
G = & \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & & {\scriptstyle 2} & & {\scriptstyle 1} & & {\scriptstyle 0} & & {\scriptstyle -1} & &
\end{array}$$

is a quasi-isomorphism, but not a homotopy equivalence.

b) Consider the map of complexes $g\colon A \to B$ given by

$$\begin{array}{rccccccccccc}
A = & \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & & \downarrow & & \downarrow & & \downarrow{\scriptstyle 2} & & \downarrow & & \\
B = & \cdots & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}/4\mathbb{Z} & \xrightarrow[2]{} & \mathbb{Z}/4\mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & & {\scriptstyle 2} & & {\scriptstyle 1} & & {\scriptstyle 0} & & {\scriptstyle -1} & &
\end{array}$$

Show that g is not nullhomotopic, but that the induced map in homology is zero.

Problem 3. A complex is *contractible* if its identity map is nullhomotopic. Show that every contractible complex of R-modules is an exact sequence.

Problem 4. Let k be a field.

a) Show that every short exact sequence of k-vector spaces splits.

b) Use part a) to prove the Rank-Nullity Theorem: given any linear transformation $T\colon V \to W$ of k-vector spaces,
$$\dim(\operatorname{im} T) + \dim(\ker T) = \dim V.$$

Problem 5. We used the Snake Lemma to deduce the long exact sequence in homology. In fact, the two statements are equivalent! Assuming the statement of the long exact sequence in homology holds, but without using the Snake Lemma, prove that any commutative diagram of R-modules with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{i} & B & \xrightarrow{p} & C & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \\
0 & \longrightarrow & A' & \xrightarrow[i']{} & B' & \xrightarrow[p']{} & C' & \longrightarrow & 0
\end{array}$$

induces an exact sequence

$$0 \longrightarrow \ker f \longrightarrow \ker g \longrightarrow \ker h \longrightarrow \operatorname{coker} f \longrightarrow \operatorname{coker} g \longrightarrow \operatorname{coker} h \longrightarrow 0.$$

Let $f\colon C \longrightarrow D$ be a map of complexes. The **mapping cone** of f is the chain complex $\operatorname{cone}(f)$ that has $C_{n-1} \oplus D_n$ in homological degree n, with differential

$$d_n := \begin{pmatrix} -d^C & 0 \\ f & d^D \end{pmatrix} : \quad \begin{array}{ccc} C_{n-1} & \xrightarrow{-d^C} & C_{n-2} \\ \oplus & \searrow^{f} & \oplus \\ D_n & \xrightarrow[d^D]{} & D_{n-1} \end{array}$$

where d^C denotes the differential on C and d^D denotes the differential on D.

Given a complex C, $\Sigma^{-1}C$ is the complex obtained by shifting C to the left and switching the sign on the differential:
$$(\Sigma^{-1}C)_n := C_{n-1} \qquad \text{and} \qquad d^{\Sigma^{-1}C} = -d^C.$$

Problem 6. Let $f\colon C \longrightarrow D$ be a map of complexes of R-modules.

a) Show that there is a short exact sequence of complexes
$$0 \longrightarrow D \xrightarrow{i} \operatorname{cone}(f) \xrightarrow{\pi} \Sigma^{-1}C \longrightarrow 0.$$

b) Consider the long exact sequence in homology induced by the previous short exact sequence. Show that the connecting homomorphism ∂ in that long exact sequence is given by the R-module homomorphisms f_n.

c) Show that f is a quasi-isomorphism if and only if $\operatorname{cone}(f)$ is an exact complex.