Final Exam

Instructions: Turn in **4** of the following problems. You cannot use any resources besides me, your classmates, and our course notes. You are not allowed to use the internet or any other textbooks as a resource.

Problem 1. Show that for all finitely generated abelian groups M and N and all $i \geqslant 2$,

$$\operatorname{Ext}^i_{\mathbb{Z}}(M, N) = 0.$$

Problem 2. Let $(R, \mathfrak{m}, k)$ be a commutative noetherian local ring, and let M be a finitely generated R-module. Show that

$$\beta_i(M) = \dim_k \left(\operatorname{Tor}_i^R(M, k)\right) = \dim_k \left(\operatorname{Ext}^i_R(M, k)\right).$$

You do not need to justify why $\operatorname{Tor}_i^R(M, k)$ and $\operatorname{Ext}^i_R(M, k)$ are k-vector spaces.

Problem 3. Show that if $\pi \colon M \to N$ is a surjective map of R-modules with M and N both flat, then $\ker \pi$ is flat.

Problem 4. Show that $\operatorname{pdim}_R M \leqslant d$ if and only if $\operatorname{Ext}^{d+1}_R(M, N) = 0$ for all R-modules N.

Problem 5. Let $(R, \mathfrak{m})$ be a commutative noetherian local ring, M and N be finitely generated R-modules, and $r \in \mathfrak{m}$. Show that if r is regular on M and $\operatorname{Ext}^i_R(M/rM, N) = 0$ for $i \gg 0$, then $\operatorname{Ext}^i_R(M, N) = 0$ for $i \gg 0$.

Hint: Show that $\operatorname{Ext}^i_R(M, N)$ is a finitely generated R-module.

Problem 6. Let $f \colon A \to B$ be a map of complexes. Show that f is nullhomotopic if and only if f factors through the canonical map $A \to \operatorname{cone}(\operatorname{id}_A)$.

Problem 7. Let $\mathcal{A}$ be an abelian category.

a) Show that $\ker(x \xrightarrow{0} y) = 1_x$, $\operatorname{coker}(x \xrightarrow{0} y) = 1_y$, and $\operatorname{im}(x \xrightarrow{0} y) = 0 \longrightarrow y$.

b) Show that f is a mono if and only if $fg = 0$ implies $g = 0$ for all g.

c) Show that f is a mono if and only if $\ker f = 0$.