Problem Set 1
Primary Decomposition and the definition of symbolic powers

Problem 1. Let $R = \mathbb{Z}[\sqrt{-5}]$. While $6 \in R$ cannot be written as a unique product of irreducibles, we are going to show that the ideal $I = (6)$ does have a unique primary decomposition. Unfortunately, Macaulay2 cannot take primary decompositions over $\mathbb{Z}$, but this one we can do the old fashioned way.

a) Prove that (2) is a primary ideal.

b) Prove that (3) is *not* a primary ideal.

c) Prove that $(3, 1+\sqrt{-5})$ and $(3, 1-\sqrt{-5})$ are both primary.

d) Show that $(6) = (2) \cap (3, 1+\sqrt{-5}) \cap (3, 1-\sqrt{-5})$.

e) Show that this primary decomposition is unique.

Problem 2. Let I and J be ideals in a noetherian ring R. Show that $I \subseteq J$ if and only if $I_P \subseteq J_P$ for all $P \in \operatorname{Ass}(J)$.

Problem 3 (2 points)**.** Let I, J, and L be ideals in a noetherian ring R.

a) There exists n such that $(I : J^\infty) = (I : J^n)$.

b) If Q is a P-primary ideal, then

$$(Q : J^\infty) = \begin{cases} Q & \text{if } J \not\subseteq P \\ R & \text{if } J \subseteq P \end{cases}.$$

c) $(I \cap J : L^\infty) = (I : L^\infty) \cap (J : L^\infty)$.

d) Given a primary decomposition $I = Q_1 \cap \cdots \cap Q_k$,

$$(I : J^\infty) = \bigcap_{J \not\subseteq \sqrt{Q_i}} Q_i.$$

e) If $\operatorname{Ass}(I) = \operatorname{Min}(I) = \{P_1, \ldots, P_k\}$, then for each i there exists an element $x_i \in R$ such that the P_i-primary component of I is given by $(I : x_i^\infty)$.

Problem 4. Let I be an ideal with no embedded primes in a noetherian ring R. Show that there exists an ideal J, which we can take to be principal, such that $I^{(n)} = (I^n : J^\infty)$ for all $n \geqslant 1$.

Problem 5 (Minimal primes and support)**.**

a) Describe $\operatorname{supp}(I/I^2)$, where $I = (xz)$ in $R = \mathbb{C}[x,y,z]/(xy, yz)$.

b) Find all the minimal primes of $J = (ab, bc, cd, ad)$ in $k[a,b,c,d]$ over any field k.

c) Find the minimal primes of the ring S, where

$$S = \mathbb{Q}\begin{bmatrix} ux & uy & uz \\ vx & vy & vz \end{bmatrix} \subseteq \frac{\mathbb{Q}[u,v,x,y,z]}{(x^3+y^3+z^3)}.$$

Fix a field k. The n**th Veronese ring in** d **variables** is the subalgebra of $R = k[x_1, \ldots, x_d]$ generated by all the degree n monomials in $x_1, \ldots, x_d$, which we will denote by $R^{(n)}$. For example, $k[x,y]^{(2)} = k[x^2, xy, y^2]$. There are $N := \binom{n+d-1}{n}$ algebra generators for $R^{(n)}$, so we can write it as a quotient of $k[y_1, \ldots, y_N]$; more precisely, the map $\pi : k[y_1, \ldots, y_N] \to R^{(n)}$ that sends each y_i to a different monomial $x_1^{a_1} \cdots x_d^{a_d}$ with $a_1 + \cdots + a_d = n$ is surjective, and if $P = \ker \pi$, $R^{(n)} \cong k[y_1, \ldots, y_N]/P$. We call this ideal P the **defining ideal** of $R^{(n)}$, though P is only defined up to the order of $y_1, \ldots, y_N$.

Problem 6. Let P be the defining ideal of $\mathbb{Q}[x,y]^{(3)}$ and Q be the defining ideal of $\mathbb{Q}[x,y,z]^{(2)}$.

a) Is P a prime ideal? Is Q a prime ideal? Why?

b) Without running any packages besides those that come preloaded with Macaulay2, find $P^{(2)}$ and $Q^{(2)}$.

c) Is $P^2 = P^{(2)}$? Is $Q^2 = Q^{(2)}$?