Problem Set 2

Problem 1. Let I be an ideal with no embedded primes in a noetherian ring R. Show that there exists an ideal J, which we can take to be principal, such that $I^{(n)} = (I^n : J^\infty)$ for all $n \geqslant 1$.

Problem 2. Let $R = \mathbb{Q}[x_1, x_2, x_3, x_4]$, and consider the ideal I you can define in Macaulay2 as follows:

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M = matrix{{0,-x_1,-x_3,x_2,-x_1,x_4,-x_3},{x_1,0,-x_3,x_2,x_1,-x_4,-x_1},
{x_3,x_3,0,0,-x_3,x_1,x_4},{-x_2,-x_2,0,0,-x_4,x_2,0},
{x_1,-x_1,x_3,x_4,0,-x_3,x_1},{-x_4,x_4,-x_1,-x_2,x_3,0,x_2},{x_3,x_1,x_4,0,-x_1,x_2,0}}
I = pfaffians(6,M)
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Using only the packages that come preloaded with Macaulay2, compute $I^{(2)}$ and $I^{(3)}$. Is $I^{(2)} = I^2$? Is $I^{(3)} = I^3$?

Problem 3. Find all the symbolic powers of $I = (6)$ in $\mathbb{Z}[\sqrt{-5}]$.

Problem 4. Let x be a regular element in a ring R, meaning that $xa = 0 \implies a = 0$.

a) Show that $(x^n : x^{n-1}) = (x)$ for all $n \geqslant 1$.

b) Show that $\operatorname{Ass}(x^n) = \operatorname{Ass}(x)$ for all $n \geqslant 1$.

Problem 5. Let $R = k[x, y, z]/(xy - z^c)$ where k is a field and $c \geqslant 2$, and let $P = (x, z)$.

a) Show that (x^n) is a primary ideal for all $n \geqslant 1$. What is its radical?

b) Prove that $P^{(cn)} = (x^n)$ for all $n \geqslant 1$.

c) Prove that $P^{(n)} \neq P^n$ for all $n \geqslant 2$.

Problem 6. Height and dimension.

a) Given any field k, find the height of $J = (ab, bc, cd, ad)$ in $R = k[a, b, c, d]$ and $\dim(R/J)$.

b) Find the dimension of the ring $S = \mathbb{Q}[x^3y^3, x^3y^2z, x^2z^3] \subseteq \mathbb{Q}[x, y, z]$.

c) Let I be the defining ideal of the curve parametrized by (t^{13}, t^{42}, t^{73}) over $\mathbb{Q}$. Find the height of I, and notice that $\operatorname{height}(I) < \mu(I)$.

d) Let $R = \mathbb{Q}[x, y, z]$, and $I = (x^3, x^2y, x^2z, xyz)$. Find the dimension of R/I and the height of I.

e) Find the dimension of the module I/I^2, where $I = (xz)$ in $R = \mathbb{C}[x, y, z]/(xy, yz)$.

Problem 7. Let k be any field.

a) Is x^2, xy, y^2 a regular sequence on $\mathbb{Q}[x^2, xy, y^2]$?

b) Is $x^3 - yz, y^2 - xz, z^2 - x^2y$ a regular sequence on $k[x, y, z]$? (Hint: you can use Example 2.4.)

c) What is the maximal possible length of a regular sequence on $R = k[\![x, y]\!]/(x^2, xy)$?

d) Let $R = k[\![x, y]\!]/(x^2, xy)$. Find a regular sequence on $M = R/(x)$ of maximal possible length.