Problem Set 5

To solve these problems, you are not allowed to use any additional Macaulay2 packages besides the `Complexes` package and the ones that are automatically loaded with Macaulay2.

Definition (Jacobian ideal)**.** *Let k be a field and $R = k[x_1, \ldots, x_n]/I$, where $I = (f_1, \ldots, f_r)$ has pure height h. The* Jacobian matrix *of R is the matrix given by*

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ & \ddots & \\ \frac{\partial f_r}{\partial x_1} & \cdots & \frac{\partial f_r}{\partial x_n} \end{pmatrix}.$$

The jacobian ideal *of R is the ideal generated by the h-minors of the Jacobian matrix.*

Turns out the Jacobian ideal is indeed well-defined – meaning, our definition does not depend on the choice of presentation for R – and that it determines the nonsmooth locus of R. For proofs of these well-known facts, see Section 16.6 of Eisenbud's *Commutative Algebra with a view towards algebraic geometry.*

Theorem (Jacobian criterion)**.** *Let $R = k[x_1, \ldots, x_n]/I$ with k a perfect field, and assume that I has pure height h. The Jacobian ideal J defines the nonsmooth locus of R: a prime P in $k[x_1, \ldots, x_n]$ contains J if and only if R_P is not a regular ring.*

Problem 1. Let $(R, \mathfrak{m})$ be a regular local ring and I be an ideal in R.

a) Show that there is a short exact sequence

$$0 \longrightarrow \frac{I+\mathfrak{m}^2}{\mathfrak{m}^2} \longrightarrow \mathfrak{m}/\mathfrak{m}^2 \longrightarrow \frac{\mathfrak{m}}{\mathfrak{m}^2+I} \longrightarrow 0$$

and conclude that $\dim_k \left(\frac{I+\mathfrak{m}^2}{\mathfrak{m}^2}\right) = \operatorname{embdim}(R) - \operatorname{embdim}(R/I)$.

b) Show that if R/I is regular, then there exists a minimal set of generators $x_1, \ldots, x_d$ for $\mathfrak{m}$ such that $I = (x_1, \ldots, x_n)$ for some n.

c) Conclude that if I is not generated by a regular sequence, then R/I is not regular.

Problem 2. Let I be a radical ideal in $R = k[x_1, \ldots, x_n]$, where k is a perfect field, and let J be the Jacobian ideal of I. Prove that $I^{(n)} = (I^n : J^\infty)$ for all $n \geqslant 1$.

Problem 3. Let k be a field, $R = k[x, y, z]$, and $I = (xy, xz, yz)$. Find the Jacobian ideal J of I, and show directly that $I^{(n)} = (I^n : J^\infty)$ for all $n \geqslant 1$ without using Problem 2.

Problem 4. For each of the following ideals I, find an **element** t such that $I^{(n)} = (I^n : t^\infty)$ for all $n \geqslant 1$.

a) $I = (xz, xw, yz, yw)$ in $R = k[x, y, z, w]$, where k is any field.

b) I is the defining ideal of the second Veronese in 3 variables $\mathbb{Q}[x, y, z]^{(2)}$.

Problem 5. Let k be a field, $R = k[x, y, z]$, and $I = (xy, xz, yz)$.

a) Show that $I^{(2n)} = \left(I^{(2)}\right)^n$ for all $n \geqslant 1$.

b) Show that $I^{(2n+1)} = I\left(I^{(2)}\right)^n$ for all $n \geqslant 1$.

c) Show that the symbolic Rees algebra of I is a noetherian ring.

Problem 6. Let $R \to S$ be a flat ring homomorphism.

a) Show that for every ideal I in R and every $x \in R$, $(I :_R x)S = (IS :_S x)$.

   Hint: what is the kernel of $R/I \xrightarrow{\cdot x} R/I$?

b) Show that I and J are ideals in R, then $IS \cap JS = (I \cap J)S$.

   Hint: what is the kernel of the canonical map $R \to R/I \oplus R/J$?

c) Show that if I and J are ideals in R with J finitely generated, then $(I :_R J)S = (IS :_S JS)$.

Problem 7. Let I be a squarefree monomial ideal in $R = k[x_1, \ldots, x_d]$, where k is a perfect field of prime characteristic p.

a) Show that R/I is F-pure.

b) Show that R/I is strongly F-regular if and only if I is generated by variables.

Problem 8. Let I be a radical ideal and fix $n \geqslant 1$. Show that $(I^n : I^{(n)})$ must contain an element not in any minimal prime of I.