Topology

Math 817

Eloísa Grifo

August 31, 2026

Warning!

Proceed with caution. These notes are under construction and are 100% guaranteed to contain typos. If you find any typos or errors, we will be most grateful to you for letting me know.

Acknowledgements

These notes take much inspiration from other sources, primarily Mark Walker's Math 871 notes from Fall 2024, Munkres' *Introduction to topology* [Mun00], and Hatcher's *Algebraic topology* [Hat02]. Some of the pictures in these notes were created with the help of ChatGPT.

Contents

Chapter 1

Topological spaces and continuous functions

1.1 Topological spaces

Definition 1.1.1. Let X be a set. A **topology** on X is a collection $\mathcal{T}$ of subsets of X, called **open sets**, satisfying the following properties:

- The empty set belongs to $\mathcal{T}$: $\emptyset \in \mathcal{T}$.
- The entire space X belongs to $\mathcal{T}$: $X \in \mathcal{T}$.
- If U and V are in $\mathcal{T}$, then so is $U \cap V$.
- If $\{U_\alpha\}_{\alpha \in I}$ is any collection of open sets in $\mathcal{T}$, then $\bigcup\limits_{\alpha \in I} U_\alpha \in \mathcal{T}$.

A **topological space** is a pair $(X, \mathcal{T})$ where $\mathcal{T}$ is a topology on the set $\mathcal{T}$.

We will often refer to an element $x \in X$ in a topological space $(X, \mathcal{T}_X)$ as a **point**.

Remark 1.1.2. We may sometimes abuse notation and say "Let X be a topological space", which should be interpreted as "Let X be a set equipped with a topology $\mathcal{T}$". Ironically, the topology $\mathcal{T}$ (meaning, the collection of open sets) is a lot more important than the set: given $\mathcal{T}$, we can recover the set X by taking the union of all the sets in $\mathcal{T}$.

Remark 1.1.3. Let $(X, \mathcal{T})$ be a topological space. Given any *finite* collection $U_1, \dots, U_n$ of open sets, applying the intersection axiom repeatedly we get that the intersection $U_1 \cap \dots \cap U_n$ is also an open set. However, note that the arbitrary intersection of open sets might not be an open set.

Definition 1.1.4. Let X be any set. The **trivial topology**, also known as the **indiscrete topology** on X, is the topology $\mathcal{T} = \{\emptyset, X\}$ consisting of only the empty set and X. The **discrete topology** on X is the topology $\mathcal{T} = \mathcal{P}(\mathcal{T})$ consisting of all the subsets of X.

One of the most important examples of a topology is the familiar standard topology on $\mathbb{R}$, or more generally on $\mathbb{R}^n$, which one may take as inspiration for everything we will do.

Notation 1.1.5. Recall that for real numbers $a < b$, by (a, b) we mean the set

$$(a, b) := \{r \in \mathbb{R} \mid a < r < b\},$$

also called the **open interval** from a to b. Similarly,

$$[a, b] := \{r \in \mathbb{R} \mid a \leqslant r \leqslant b\}$$

is the **closed interval** from a to b. We may also write $[a, b)$ or $(a, b]$ for the intervals that include a but not b, or b but not a, respectively.

Example 1.1.6. The **Euclidean topology** or **standard topology** on the set of real numbers $\mathbb{R}$ is defined as follows: a subset U of $\mathbb{R}$ is *open* if for all $x \in U$ there is a positive real number $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \subseteq U$. Note that every nonempty open set in this topology is infinite.

In more informal terms, U is open if and only if every point in U is a strictly positive distance away from all points not in U. In particular, for all real numbers $a < b$, the set (a, b) is an open set in the standard topology for $\mathbb{R}$. Let us prove this carefully.

Let $x \in (a, b)$. Note that $a < x < b$, so $\varepsilon := \min\{b - x, x - a\} > 0$. We claim that $(x - \varepsilon, x + \varepsilon) \subseteq (a, b)$. To see this, pick $r \in (x - \varepsilon, x + \varepsilon)$. Since $\varepsilon \leqslant x - a$, we have $x - \varepsilon \geqslant a$, and hence $r > x - \varepsilon \geqslant a$ gives $r > a$. A similar argument shows that since $\varepsilon \leqslant b - x$, we have $r < b$. This proves that $r \in (a, b)$. Therefore, since r was arbitrary, we have $(x - \varepsilon, x + \varepsilon) \subseteq (a, b)$. Since this holds for any $x \in (a, b)$, this shows that (a, b) is open.

Remark 1.1.7. Note that while the standard topology on $\mathbb{R}$ is the one that most matches our natural intuition, the set $\mathbb{R}$ can be given other topologies; for example, we can consider the discrete topology or the trivial topology on $\mathbb{R}$. In fact, given any set X, the term *open subset* depends on the topology we are considering, and it is not in any sense intrinsic to X.

But of course, the standard topology plays a special role in many other areas of mathematics, so it is worth keeping it in mind for some guidance and intuition. In fact, there is also a standard topology on $\mathbb{R}^n$:

Example 1.1.8. The **Euclidean topology** or **standard topology** on $X = \mathbb{R}^n$ is given as follows: a subset U of X is open if for every $v \in U$ there exists a positive real number $\varepsilon > 0$ such that $B_\varepsilon(v) \subseteq X$, where

$$B_\varepsilon(v) = \{w \in \mathbb{R}^n \mid \operatorname{dist}(v, w) < \varepsilon\}$$

and dist denotes the Euclidean distance between v and w. In other words, $B_\varepsilon(v)$ denotes the open ball of radius ε centered at v. One can adapt the proof we gave in Example 1.1.8 to show that this is indeed a topology. Note in particular that every nonempty open set in this topology is infinite.

Example 1.1.9. We can now give an explicit example showing that the arbitrary intersection of open sets if not necessarily open. In $\mathbb{R}$ with the standard topology, the intersection of open sets

$$\bigcap_{n \geqslant 1} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$$

cannot be an open set, since it is finite.

Example 1.1.10. Let $X = \{a, b, c\}$ be an element with three elements, and consider

$$\mathcal{T} = \{\emptyset, X, \{a, b\}, \{b\}, \{b, c\}\}.$$

One check (however tedious that may be) that $\mathcal{T}$ is in fact a topology on X.

Example 1.1.11 (The Finite Complement Topology)**.** Let X be any set. The **finite complement topology** on X is the topology

$$\mathcal{T} = \{U \subseteq X \mid X \setminus U \text{ is finite}\} \cup \{\emptyset\}$$

where the open sets are the empty set and all the sets that contain all but a finite number of elements of X.

Exercise 1.1.12. Check that the finite complement topology is indeed a topology.

We will often be concerned with topologies on $\mathbb{R}$ or $\mathbb{R}^n$, so we will make use of some of the nice properties of the real numbers. One such property we will use is the **Completeness Axiom** of the real numbers:

Every nonempty, bounded above subset of $\mathbb{R}$ *has a least upper bound.*

Let us properly recall what all these terms mean:

Definition 1.1.13. A subset $X \subseteq \mathbb{R}$ is **bounded above** if there exists $u \in \mathbb{R}$ such that

$$x \leqslant u \quad \text{for all } x \in X.$$

Any u with this property is an **upper bound** for X. A subset $X \subseteq \mathbb{R}$ is **bounded below** if there exists $\ell \in \mathbb{R}$ such that

$$\ell \leqslant x \quad \text{for all } x \in X.$$

Any ℓ with this property is a **lower bound** for X. A **least upper bound** for a subset $X \subseteq \mathbb{R}$ is a real number $u \in \mathbb{R}$ such that u is an upper bound for X and $u \leqslant w$ for all upper bounds w for X. A **greatest lower bound** for a subset $X \subseteq \mathbb{R}$ is a real number $\ell \in \mathbb{R}$ such that ℓ is an upper bound for X and $w \leqslant \ell$ for all lower bounds w for X.

Example 1.1.14. The **infinite ray topology** on $\mathbb{R}$ is given by

$$\mathcal{T} = \{(-\infty, a) \mid a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}.$$

We can think of $\emptyset$ and $\mathbb{R}$ as informally arising by taking $a = -\infty$ and $a = \infty$, respectively, though this is not formally correct (since these are not elements of $\mathbb{R}$).

Exercise 1.1.15. Show that the infinite ray topology on $\mathbb{R}$ is indeed a topology.
Hint: you may need to use the Completeness Axiom for $\mathbb{R}$.

Exercise 1.1.16. Is the following a topology on $\mathbb{R}$?

$$\mathcal{T} = \{(-\infty, a] \mid a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$$

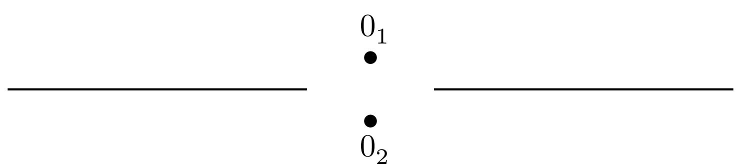
Example 1.1.17 (The line with two origins)**.** Let L be the real line with two rival origins; more precisely, we add in a point to the real line, and call our two origins 0_1 and 0_2:

$$L = (\mathbb{R} \setminus \{0\}) \cup \{0_1, 0_2\}.$$

If one removes 0_2 from L, we get the set $\mathbb{R}_1 = (\mathbb{R} \setminus \{0\}) \cup \{0_1\}$, which will identify with $\mathbb{R}$ itself in the natural way, by identifying 0_1 with 0. Similarly, we may identify $\mathbb{R}_2 = X \setminus \{0_1\} = (\mathbb{R} \setminus \{0\}) \cup \{0_2\}$ with $\mathbb{R}$. The **line with two origins** is the topological space with underlying set L where a subset U of L is open if $U \setminus \{0_2\}$ is an open subset of $\mathbb{R}_1$ equipped with the standard topology and $U \setminus \{0_1\}$ is an open subset of $\mathbb{R}_2$ equipped with the standard topology. We claim that the line with two origins forms a topology on X, which we leave as an exercise.

Let U be any open set in L containing 0_1. Then $U \cap \mathbb{R}_1$ is an open set in the standard topology of $\mathbb{R}$, so there exists $\varepsilon > 0$ such that $(-\varepsilon, \varepsilon) \subseteq U \cap \mathbb{R}_1$. Then in L, $(-\varepsilon, 0) \cup \{0_1\} \cup (0, \varepsilon) \subseteq U$. Analogously, for any open set V in L containing 0_2 there exists $\delta > 0$ such that $(-\delta, 0) \cup \{0_1\} \cup (0, \delta) \subseteq V$. In particular, this shows that if $U \ni 0_1$ and $V \ni 0_2$ are open subsets of L, then $U \cap V$ is nonempty. That is, one cannot *separate* the two origins from each other by open subsets. We will come back to this point later and understand it better.

Here is a depiction of the line with two origins:



Exercise 1.1.18. Show that the line with two origins is a topological space.

Here are a few other general constructions that produce a topology for any set:

Example 1.1.19. Let X be a nonempty set and fix a point $a \in X$. The **included point topology** on X with included point a is the topology given by

$$\mathcal{T} = \{U \subseteq X \mid a \in U\} \cup \{\emptyset\}.$$

The **excluded point topology** on X with excluded point a is the topology

$$\mathcal{T} = \{U \subseteq X \mid a \notin U\} \cup \{X\}.$$

Exercise 1.1.20. Let $(X, \mathcal{T})$ be a topological space, and let A be a subset of X. Suppose that for every $x \in A$, there exists an open set U such that $x \in U \subseteq A$. Show that A is an open set.

Definition 1.1.21. Let $(X, \mathcal{T}_X)$ be a topological space. An **open neighborhood**, or simply **neighborhood**, of $x \in X$ is any open subset U such that $x \in U$.

Many statements in topology are best phrased in terms of neighborhoods, as we will soon see.

Exercise 1.1.22. Let X be any nonempty set and fix a point $a \in X$. Show that the included point topology and the excluded point topology on X with point a are both in fact topologies.

Given two different topologies on the same set, we may compare them.

Definition 1.1.23. Let X be a set and let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on X. We say that $\mathcal{T}$ is **finer** than $\mathcal{T}'$, or equivalently that $\mathcal{T}'$ is **coarser** than $\mathcal{T}$, if $\mathcal{T}' \subseteq \mathcal{T}$.

Example 1.1.24. Given any set X, the trivial topology is coarser than the discrete topology, and the discrete topology is finer than the trivial topology.

Example 1.1.25. In $\mathbb{R}$, the trivial topology is coarser than the standard topology, and the discrete topology is finer than the standard topology.

Problem 1.1.26. Let X be a set and let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on X. We say that $\mathcal{T}$ is **finer** than $\mathcal{T}'$, or equivalently that $\mathcal{T}'$ is **coarser** than $\mathcal{T}$, if $\mathcal{T}' \subseteq \mathcal{T}$. Note that two topologies can also be **incomparable**. Compare the following topologies on $\mathbb{R}$:

a) $\mathcal{T}_1$ the standard topology.

b) $\mathcal{T}_2$ the finite complement topology.

c) $\mathcal{T}_3 = \{(-\infty, a) \mid a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$ the infinite ray topology.

d) $\mathcal{T}_4 = \{(a, \infty) \mid a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$.

e) $\mathcal{T}_5$ the discrete topology.

f) $\mathcal{T}_6$ the indiscrete topology.

1.2 Bases

A basis for a topology on X is a collection of subsets of X that naturally determines a topology.

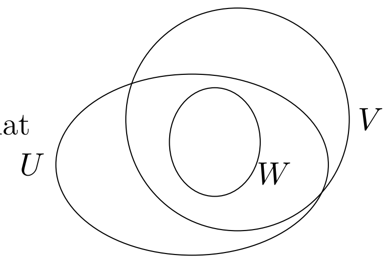
Definition 1.2.1. Let X be a set. A **basis** for a topology $\mathcal{T}$ on X consists of a collection of subsets of X, $\mathscr{B} \subseteq \mathcal{P}(X)$, with the following properties:

- For every $x \in X$, there exists $B \in \mathscr{B}$ such that $x \in B$. Equivalently,

$$\bigcup_{B \in \mathscr{B}} B = X.$$

- For all $U, V \in \mathscr{B}$, if $x \in U \cap V$, there exists $W \in \mathscr{B}$ such that

$$x \in W \subseteq U \cap V.$$



Remark 1.2.2. Note that the second condition in the definition of a basis is automatically satisfied if the intersection of two basis elements is again in the basis. However, as we will soon see, that is not what happens in general.

Definition 1.2.3. Given a basis $\mathscr{B}$ for a topology on X, the **topology generated by** $\mathscr{B}$ is the topology whose open sets are the subsets $U \subseteq X$ with the following property: for every $x \in U$, there exists $B \in \mathscr{B}$ such that $x \in B \subseteq U$.

Remark 1.2.4. Note that if $\mathcal{T}$ is the topology generated by the basis $\mathscr{B}$, each element $B \in \mathscr{B}$ is itself an open set in $\mathcal{T}$.

Lemma 1.2.5. *Let X be a set and $\mathscr{B}$ be a basis for a topology on X. Then the topology generated by $\mathscr{B}$ is indeed a topology on X.*

Proof. Let $\mathcal{T}$ be the topology generated by $\mathscr{B}$. The condition for $\emptyset \in \mathcal{T}$ is vacuously satisfied. Moreover, by definition of basis, for every $x \in X$ there exists $U \in \mathscr{B}$ such that $x \in U (\subseteq X)$, so $X \in \mathcal{T}$ by definition of $\mathcal{T}$.

Suppose that $A, B \in \mathcal{T}$, and let us show that $A \cap B \in \mathcal{T}$. Pick any $x \in A \cap B$. Then since $A \in \mathcal{T}$, by definition there exists $U \in \mathscr{B}$ such that $x \in U \subseteq A$ and, since $B \in \mathcal{T}$, there exists $V \in \mathscr{B}$ such that $x \in V \subseteq B$. By definition of a basis, there exists $W \in \mathscr{B}$ such that $x \in W \subseteq U \cap V \subseteq A \cap B$. This shows that $A \cap B \in \mathcal{T}$.

Consider an arbitrary collection of $U_\alpha \in \mathcal{T}$ indexed by a set I. Set

$$U = \bigcup_{\alpha \in I} U_\alpha.$$

To show that $U \in \mathcal{T}$, consider $x \in U$. Then $x \in U_\alpha$ for some $\alpha \in I$, and since $U_\alpha \in \mathcal{T}$, there exists $B_\alpha \in \mathscr{B}$ such that $x \in B_\alpha \subseteq U_\alpha \subseteq U$. Since this holds for all $x \in U$, we have shown that $U \in \mathcal{T}$. $\square$

Definition 1.2.6. Given a basis $\mathscr{B}$ for a topology on X, the elements of $\mathscr{B}$ are called **basic open sets**.

Definition 1.2.7. Let $(X, \mathcal{T})$ be a topological space. A **basis** for the topology $\mathcal{T}$ on X is a basis for a topology on X that generates the topology $\mathcal{T}$.

Remark 1.2.8. Every topology has a basis. Indeed, if $(X, \mathcal{T})$ is a topological space, then $\mathcal{T}$ is a basis for itself.

However, we may be interested in smaller, more tractable bases. Once we have a basis for a given topology, we can reconstruct all open sets from the basic open sets: each open set is in fact a union of basic open sets.

Theorem 1.2.9. *Let $\mathscr{B}$ be a basis for a topology on a set X and let $\mathcal{T}$ be the topology generated by $\mathscr{B}$. A subset U of X is open for the topology $\mathcal{T}$ if and only if U is an arbitrary union of basic open sets in $\mathscr{B}$.*

Proof. ($\Leftarrow$) By definition, each member of $\mathscr{B}$ is open (meaning it belongs to $\mathcal{T}$). Therefore, by definition of topology any arbitrary union of basic open sets in $\mathscr{B}$ must be open.

($\Rightarrow$) Let U be open in the topology $\mathcal{T}$. By definition of the topology $\mathcal{T}$ generated by $\mathscr{B}$, for each $x \in U$ there is a basic open $B_x \in \mathscr{B}$ such that $x \in B_x \subseteq U$. Therefore,

$$U = \bigcup_{x \in U} B_x. \qquad \square$$

Example 1.2.10. Let $\mathscr{B}$ be the collection of nonempty open intervals of $\mathbb{R}$:

$$\mathscr{B} := \{(a, b) \mid a < b\}.$$

We claim that $\mathscr{B}$ is a basis for a topology on $\mathbb{R}$. First, given any $x \in \mathbb{R}$ we can always find $a, b \in \mathbb{R}$ such that $a < x < b$, so $x \in (a, b)$. Moreover, given any two (a, b) and (c, d) in $\mathscr{B}$, if $x \in (a, b) \cap (c, d)$ then $c < x < b$ and in particular $c < b$. Similarly, $a < d$. Set

$$r = \max\{a, c\} \qquad \text{and} \qquad s = \min\{b, d\}.$$

Note that $r < x < s$. Then $x \in (r, s) \subseteq (a, b) \cap (c, d)$.

Remark 1.2.11. Example 1.2.10 is misleadingly simple, since in this case $(a, b) \cap (c, d)$ is another open interval whenever it is nonempty. In general, it need not be the case that the intersection of any two members basis open sets is either empty or a member of the basis, as we will see in Example 1.2.12.

Example 1.2.12. In $\mathbb{R}^n$, the collection of all open balls

$$B_\varepsilon(v) = \{w \in \mathbb{R}^n \mid \operatorname{dist}(v, w) < \varepsilon\}$$

with all centers and all radii $\varepsilon > 0$ forms a basis for the standard topology. When $n = 1$, note that this is precisely the basis we gave in Example 1.2.10. When $n \geqslant 2$, this is a bit more interesting. In particular, note that two balls with different centers can have a nontrivial intersection which is no longer a ball.

Note that the same topological space may have different bases.

Example 1.2.13. We claim that the collection

$$\mathscr{B} = \{(a, b) \mid a, b \in \mathbb{Q}\}$$

of open intervals *with rational endpoints* is a basis for the standard topology on $\mathbb{R}$. First, one can repeat the argument in Example 1.2.10 to show that $\mathscr{B}$ is indeed a basis for a topology on $\mathbb{R}$. Note in particular that this basis $\mathscr{B}$ is contained in the basis for the standard topology we gave in Example 1.2.10, so the topology generated by $\mathscr{B}$ is necessarily contained in the standard topology. It thus suffices to show that if U is an open set in the standard topology, then U is open in the topology generated by $\mathscr{B}$.

Let $x \in U$. By definition, there exists $\varepsilon > 0$ such that $(x-\varepsilon, x+\varepsilon) \subseteq U$. By the Density of the Rationals in $\mathbb{R}$, we can find rational numbers $q, r \in \mathbb{Q}$ such that $x - \varepsilon < q < x$ and $x < r < x + \varepsilon$. Therefore, $x \in (q, r) \in \mathscr{B}$. Note that in fact

$$x \in (q, r) \subseteq (x - \varepsilon, x + \varepsilon) \subseteq U.$$

By definition, this shows that U is in the topology generated by $\mathscr{B}$. Finally, we conclude that the topology generated by $\mathscr{B}$ is the standard topology.

Remark 1.2.14. Example 1.2.13 shows that there is a *countable* basis for the standard topology on $\mathbb{R}$. This will turn out to be a very useful fact.

Exercise 1.2.15. Let X be any nonempty set. Show that the collection of singletons (one-point subsets) of X is a basis for the discrete topology on X.

Example 1.2.16. In $\mathbb{R}$, consider

$$\mathscr{B} = \{[a, b) \mid a < b\}.$$

Note that the intersection of two elements of $\mathscr{B}$ is either empty or also an element of $\mathscr{B}$. Moreover, given $[a, b)$ and $[c, d)$ in $\mathscr{B}$, assume without loss of generality that $a \leqslant c$. If $x \in [a, b) \cap [c, d)$ then in particular $[a, b) \cap [c, d)$ is nonempty, and thus $c \leqslant x < b$. Hence $c < b$ and

$$x \in [x, b) \subseteq [c, b) \subseteq [a, b) \cap [c, d).$$

This shows that $\mathscr{B}$ is basis for a topology on $\mathbb{R}$, which is known as the **lower limit topology**. The topological space given by $\mathbb{R}$ with the lower limit topology is sometimes denoted by $\mathbb{R}_\ell$. Note that a subset of $\mathbb{R}_\ell$ is open if and only if it is a union of half-open intervals of the form $[a, b)$.

Exercise 1.2.17. Show that the lower limit topology is strictly finer than the standard topology on $\mathbb{R}$.

Exercise 1.2.18. Let $\mathscr{B}_1$ and $\mathscr{B}_2$ be bases for a topology on X. Show that if $\mathscr{B}_1 \subseteq \mathscr{B}_2$, then the topology generated by $\mathscr{B}_2$ is finer than the topology generated by $\mathscr{B}_1$.

Exercise 1.2.19. Let $\mathscr{B}$ and $\mathscr{B}'$ be bases for topologies on the same set X, and let $\mathcal{T}$ and $\mathcal{T}'$ be the topologies generated by $\mathscr{B}$ and $\mathscr{B}'$ respectively. Show that the following are equivalent:

a) The topology $\mathcal{T}'$ is finer that $\mathcal{T}$.

b) For each $x \in X$ and $B \in \mathscr{B}$ with $x \in B$, there exists $B' \in \mathscr{B}'$ such that $x \in B' \subseteq B$.

Example 1.2.20. We can now compare the lower limit topology on $\mathbb{R}$ to the standard topology. Given any point $x \in \mathbb{R}$ and any basis element $B = (a, b)$ for the standard topology such that $x \in B$, note that we necessarily have $a < x < b$, so $B' = [x, b)$ is a basic open set for the lower limit topology and

$$x \in B' \subseteq B.$$

By Exercise 1.2.19, the lower limit topology $\mathbb{R}_\ell$ is finer than the standard topology on $\mathbb{R}$.

On the other hand, these two topologies do *not* coincide: for example, $[0, 1)$ is a basic open set in the lower limit topology with $0 \in [0, 1)$, but there are no open sets in the standard topology of $\mathbb{R}$ that contain 0 and are contained in $[0, 1)$. Thanks to Exercise 1.2.19, it even suffices to check that there are no open intervals (a, b) containing 0 and contained in $[0, 1)$.

We can now give an example of two incomparable topologies on the same set:

Exercise 1.2.21. Let

$$K := \left\{ \frac{1}{n} \mid n \in \mathbb{Z}, n \geqslant 1 \right\}.$$

Show that

$$\{(a, b) \mid a < b\} \cup \{(a, b) \setminus K \mid a < b\}$$

is a basis for a topology on $\mathbb{R}$, known as the **K-topology** on $\mathbb{R}$. The topological space given by $\mathbb{R}$ with the K-topology is denoted by $\mathbb{R}_K$.

Exercise 1.2.22. Show that the topological spaces $\mathbb{R}_\ell$ and $\mathbb{R}_K$ are incomparable.

Example 1.2.23. Let $\mathscr{B}$

$$\mathscr{B} = \{(a, b) \times (c, d) \mid a < b, c < d\}$$

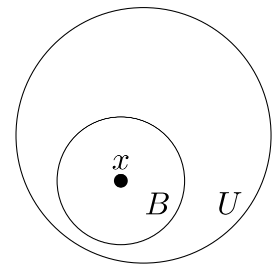
be the collection of open rectangles in $\mathbb{R}^2$. Any point in $\mathbb{R}^2$ belongs to some open rectangle, and the intersection of two open rectangles is an open rectangle, so this is a basis for a topology on $\mathbb{R}^2$, which we will temporarily call the rectangle topology. Let us compare it to the standard topology on $\mathbb{R}^2$.

We saw in Example 1.2.12 that the the set of open balls is a basis for the standard topology on $\mathbb{R}^2$. Given any point x in $\mathbb{R}^2$ and any rectangle L containing it, any open ball $B = B_\varepsilon(x)$ centered at x with radius ε small enough will be contained in L. Thus by Exercise 1.2.19, the standard topology is finer than the rectangle topology. Conversely, given any point x and any open ball containing it, we can find an open rectangle containing x that is small enough to be contained in the ball.[1] We conclude that in fact the two topologies are the same, and hence we will no longer refer to it as the rectangle topology.

[1] Think about how could make this argument more precise. Perhaps start by taking the smallest distance of x to a point in the boundary circle of the ball, and think of a precise rectangle we could define that contains x and is contained in our circle.

This example illustrates a useful way to check that a collection of open sets is actually a basis for a given topology:

Lemma 1.2.24. *Let $(X, \mathcal{T})$ be a topological space. Suppose $\mathscr{B}$ is a collection of open subsets of X (meaning that $\mathscr{B} \subseteq \mathcal{T}$) such that the following condition holds:*



for every open set U and every $x \in U$, there exists $B \in \mathscr{B}$ such that

$$x \in B \subseteq U.$$

Then $\mathscr{B}$ is a basis for the topology $\mathcal{T}$.

Proof. We have to check two things: that $\mathscr{B}$ is a basis and that the topology generated by $\mathscr{B}$ coincides with $\mathcal{T}$. First, we prove that $\mathscr{B}$ is in fact a basis for a topology. Given $x \in X$, since X is open, by assumption there exists $B \in \mathscr{B}$ such that $x \in B \subseteq X$. In particular,

$$\bigcup_{B \in \mathscr{B}} B = X.$$

Now given $B_1, B_2 \in \mathscr{B}$ and $x \in B_1 \cap B_2$, note that B_1 and B_2 are by assumption open sets, so $B_1 \cap B_2$ is open, and thus by assumption there exists $B_3 \in \mathscr{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$. Thus $\mathscr{B}$ is a basis for a topology on X. Let $\mathcal{T}'$ be the topology generated by $\mathscr{B}$.

Now note that $\mathcal{T}$ is a basis for itself. Since $\mathscr{B}$ is a basis for $\mathcal{T}'$, and $\mathscr{B} \subseteq \mathcal{T}$, Exercise 1.2.18 applies and implies that $\mathcal{T}' \subseteq \mathcal{T}$.

Let U be any open set in $\mathcal{T}$. By assumption, for each $x \in U$ there exists $B_x \in \mathscr{B}$ such that $x \in B_x \subseteq U$. In particular,

$$U = \bigcup_{x \in U} \{x\} \subseteq \bigcup_{x \in U} B_x \subseteq U,$$

so U is a union of basic open sets in $\mathscr{B}$. By Theorem 1.2.9, U is an open set in $\mathcal{T}'$, the topology generated by $\mathscr{B}$. We conclude that $\mathcal{T} \subseteq \mathcal{T}'$ and thus the two topologies coincide. □

Remark 1.2.25. We can now rewrite Lemma 1.2.24 in a nice compact way: a collection of open sets $\mathscr{B}$ is a basis for the topology on X if and only if for all $x \in X$, every neighborhood of x contains a neighborhood of X contained in $\mathscr{B}$.

Definition 1.2.26. A **subbasis** for a topology on a set X is a collection $S \subseteq \mathcal{P}(X)$ of subsets of X such that

$$\bigcup_{U \in S} U = X.$$

Exercise 1.2.27. Let S be a subbasis for a topology on S. Show that the collection $\mathscr{B}$ of all finite intersections of elements of S is a basis for a topology on X. The topology generated by $\mathscr{B}$ is the **topology generated by** S.

Remark 1.2.28. The topology generated by a subbasis S is the smallest topology containing S.

1.3 The product topology for finite products

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Our goal is to equip the cartesian product

$$X \times Y = \{(x, y) \mid x \in X, y \in Y\}$$

with a reasonable topology.

Theorem 1.3.1. *Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. The set*

$$\mathscr{B} := \{U \times V \mid U \in \mathcal{T}_X \text{ and } V \in \mathcal{T}_Y\}$$

is a basis for a topology on $X \times Y$, called the **product topology**. *More generally, if $\mathscr{B}_X$ is a basis for $\mathcal{T}_X$ and $\mathscr{B}_Y$ is a basis for $\mathcal{T}_Y$, then*

$$\mathscr{B}_{X\times Y} := \{U \times V \mid U \in \mathscr{B}_X \text{ and } V \in \mathscr{B}_Y\}$$

is also a basis for the product topology on $X \times Y$.

Proof. Let $(x, y) \in X \times Y$. By definition of basis, there exist $U \in \mathscr{B}_X$ and $V \in \mathscr{B}_Y$ such that $x \in U$ and $y \in V$. Therefore, $(x, y) \in U \times V$ with $U \times V \in \mathscr{B}_{X\times Y}$. This proves the first axiom for a basis is satisfied.

Suppose $(x, y) \in U \times V \cap U' \times V'$, where $U, U' \in \mathscr{B}_X$ and $V, V' \in \mathscr{B}_Y$. Then $x \in U \cap U'$ and $y \in V \cap V'$. Since $\mathscr{B}_X$ and $\mathscr{B}_Y$ are bases for $\mathcal{T}_X$ and $\mathcal{T}_Y$, respectively, there exists $U'' \in \mathscr{B}_X$ such that

$$x \in U'' \subseteq U \cap U'.$$

Analogously, there exists $V'' \in \mathscr{B}_Y$ with

$$y \in V'' \subseteq V \cap V'.$$

Therefore,

$$(x, y) \in U'' \times V'' \subseteq U \times V \cap U' \times V'.$$

This concludes the proof that $\mathscr{B}_{X\times Y}$ is a basis for a topology on $X \times Y$, proving the second part of our statement. It remains to show that $\mathscr{B}$ and $\mathscr{B}_{X\times Y}$ are basis for the same topology.

Now note that we can take $\mathscr{B}_X = \mathcal{T}_X$ and $\mathscr{B}_Y = \mathcal{T}_Y$ as bases for each of the topologies on X and Y, respectively, which proves $\mathscr{B}$ is also a basis for a topology on $X \times Y$. Note that

$$\mathscr{B}_{X\times Y} \subseteq \mathscr{B},$$

so by Exercise 1.2.18 the topology generated by $\mathscr{B}$ is finer than the product topology.

Conversely, fix $(x, y) \in X \times Y$ and $B \in \mathscr{B}$, and say $B = U \times V$ for some $U \in \mathcal{T}_X$ and $V \in \mathcal{T}_Y$. Since $\mathscr{B}_X$ and $\mathscr{B}_Y$ are bases for the respective topologies, we can find $B_x \in \mathscr{B}_X$ and $B_y \in \mathscr{B}_Y$ such that such that

$$x \in B_x \subseteq U \quad \text{and} \quad y \in B_y \subseteq V.$$

Then

$$(x, y) \in B_x \times B_y \subseteq U \times V.$$

By Exercise 1.2.19, the product topology is finer than the topology generated by $\mathscr{B}$. We conclude that the two bases generated the same topology. □

Remark 1.3.2. In general, given topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$, the set

$$\{U \times V \mid U \in \mathcal{T}_X \text{ and } V \in \mathcal{T}_Y\}$$

is not a topology on $X \times Y$, and we truly need to consider unions of such sets.

Example 1.3.3. We saw in Example 1.2.10 that the set of open intervals forms a basis for the standard topology on $\mathbb{R}$, so the set of open rectangles forms a basis for the product topology on $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$. On the other hand, by Example 1.2.23 this basis generates the standard topology on $\mathbb{R}^2$. In summary: the product topology on $\mathbb{R}^2$ agrees with the standard topology.

This example illustrates Remark 1.3.2 perfectly: there are open sets in $\mathbb{R}^2$ that are not open rectangles.

Exercise 1.3.4. Consider $X = \mathbb{R}$ with the standard topology and $Y = \{0, 1\}$ with the discrete topology. Describe the product topology on $X \times Y$, by describing what any arbitrary open set looks like.

Remark 1.3.5. One can of course iterate this process. Given topological spaces $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$, and $(Z, \mathcal{T}_Z)$, we may consider $(X \times Y) \times Z$ and $X \times (Y \times Z)$, and take the product topologies on each. Note that as sets, these are technically not the same set: one has elements of the form $(x, (y, z))$ and the other elements of the form $((x, y), z)$. But there is of course a bijection between the two and $X \times Y \times Z$, whose elements are triples of the form (x, y, z) with $x \in X$, $y \in Y$, and $z \in Z$. We then consider the natural product topology, with basis

$$\{U \times V \times W \mid U \in \mathscr{B}_X, V \in \mathscr{B}_Y, W \mathscr{B}_Z\}$$

given any bases $\mathscr{B}_X$, $\mathscr{B}_Y$, and $\mathscr{B}_Z$ for the respective topological spaces. One can check that in fact this matches the definition of the product topology on $(X \times Y) \times Z$ and $X \times (Y \times Z)$.

Definition 1.3.6. Given topological spaces $(X_1, \mathcal{T}_1), \cdots, (X_n, \mathcal{T}_n)$, the **product topology** on $X_1 \times \cdots \times X_n$ is the topology generated by the basis

$$\{U_1 \times \cdots \times U_n \mid U_i \in \mathcal{T}_i\}.$$

One can also define a product topology on an arbitrary product of spaces. However, that carries with it some interesting new challenges, and it turns out there is more than one good choice of how to naturally define a topology on an arbitrary product of spaces. We will discuss this in more detail later.

Exercise 1.3.7. Let $\mathbb{R}^n$ denote $\overbrace{\mathbb{R} \times \cdots \times \mathbb{R}}^{n}$. Show that the product topology on

$$\mathbb{R}^n = \overbrace{\mathbb{R} \times \cdots \times \mathbb{R}}^{n},$$

where each factor $\mathbb{R}$ is given the standard topology, coincides with the standard topology in $\mathbb{R}^n$.

Exercise 1.3.8. Consider the space $\mathbb{R}_\ell$ consisting of $\mathbb{R}$ equipped with the lower limit topology, and let $\mathbb{R}$ be the real line equipped with the standard topology. Describe bases for $\mathbb{R}_\ell \times \mathbb{R}$ and $\mathbb{R}_\ell \times R_\ell$.

1.4 The subspace topology

Theorem 1.4.1. *Let $(X, \mathcal{T}_X)$ be a topological space and Y any subset of X. Set*

$$\mathcal{T}_Y := \{U \cap Y \mid U \in \mathcal{T}_X\}.$$

Then $\mathcal{T}_Y$ is a topology on Y.

Proof. Since $\emptyset \in \mathcal{T}_X$ and $X \in \mathcal{T}_X$,

$$\emptyset = \emptyset \cap Y \in \mathcal{T}_Y \quad \text{and} \quad Y = X \cap Y \in \mathcal{T}_Y.$$

Fix U and V in $\mathcal{T}_Y$, and let $A, B \in \mathcal{T}_X$ be such that $U = A \cap Y$ and $V = B \cap Y$. Since $\mathcal{T}_X$ is a topology, $A \cap B \in \mathcal{T}_X$. Therefore,

$$U \cap V = (A \cap Y) \cap (B \cap Y) = \underbrace{(A \cap B)}_{\in \mathcal{T}_X} \cap Y.$$

Finally, given a collection $\{U_\alpha\}$ of elements of $\mathcal{T}_Y$ indexed by a set Λ, where each U_α is of the form $U_\alpha = V_\alpha \cap Y$ for some open set V_α in X, since $\mathcal{T}_X$ is a topology we have

$$\bigcup_{\alpha \in \Lambda} V_\alpha \in \mathcal{T}_X.$$

Thus

$$\bigcup_{\alpha \in \Lambda} U_\alpha = \bigcup_{\alpha \in \Lambda} (V_\alpha \cap Y) = \underbrace{\left(\bigcup_{\alpha \in \Lambda} V_\alpha \right)}_{\in \mathcal{T}_X} \cap Y \in \mathcal{T}_Y.$$

This finishes the proof that $\mathcal{T}_Y$ is a topology. $\square$

Definition 1.4.2. Given a topological space $(X, \mathcal{T}_X)$, the **subspace topology** on $Y \subseteq X$ is the topology with open sets

$$\mathcal{T}_Y := \{U \cap Y \mid U \in \mathcal{T}\}.$$

Remark 1.4.3. Suppose Y itself is an open subset of X, equipped with the subspace topology. Then given any open subset U of X, $U \cap Y$ is open in Y (by definition), but also open in X since it is an intersection of open sets (of X). Thus whenever Y itself is open in X, the open subsets of Y for the subspace topology coincide with those open subsets of X that happen to be contained in Y.

Exercise 1.4.4. Identify $\mathbb{R}$ with $\mathbb{R} \times \{0\} \subseteq \mathbb{R}^2$ in the natural way. Show that the subspace topology on $\mathbb{R}$ coincides with the standard topology.

Before we give more examples, the following unsurprising lemma will be helpful:

Lemma 1.4.5. *Let $\mathscr{B}_X$ be a basis for the topology on a topological space $(X, \mathcal{T}_X)$, and Y is any subset of X. Then*

$$\mathscr{B}_Y := \{U \cap Y \mid U \in \mathscr{B}_X\}$$

is a basis for the subspace topology on Y.

Proof. First, note that every element in $\mathscr{B}_Y$ is an open set in the subspace topology of Y, so we can apply Lemma 1.2.24. Any open subset of Y has the form $U \cap Y$ for some open set U in $\mathcal{T}_X$. Let $y \in U \cap Y$. Since U is open in X and $\mathscr{B}$ is a basis for $\mathcal{T}_X$, there exists $B \in \mathscr{B}_X$ such that $x \in B \subseteq U$. Therefore, $y \in B \cap Y \subseteq U \cap Y$. By Lemma 1.2.24, $\mathscr{B}_Y$ is a basis for the subspace topology on Y. $\square$

Example 1.4.6. Consider $\mathbb{R}$ with the standard topology, and endow $[0, 1]$ with the subspace topology. Since the collection of open intervals is a basis for the topology on $\mathbb{R}$, by Lemma 1.4.5 we can obtain a basis for the subspace topology on $[0, 1]$ by intersecting all open intervals on $\mathbb{R}$ with $[0, 1]$. We conclude that

$$\{(a, b) \mid 0 < a < b < 1\} \cup \{[0, b) \mid 0 < b < 1\} \cup \{(a, 1] \mid 0 < a < 1\}$$

is a basis for the subspace topology on $[0, 1]$. Note that some of the open sets in $[0, 1]$ do not remain open when we consider them as subsets of $\mathbb{R}$: for example, $(0, 1]$ is open in $[0, 1]$ but not in $\mathbb{R}$.

Remark 1.4.7. **??** and **??** shows that when Y is a subspace of a topological space X and we want to talk about a subset U of Y, there is an important distinction between *open in* Y and *open in* X, and thus we should always avoid simply saying *open* without indicating *in what space*. Note that if $U \subseteq Y \subseteq X$ is open in X, then it is also open in Y in the subspace topology, but as **??** and **??** shows, U might be open in Y but not in X.

Lemma 1.4.8. *Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, and $A \subseteq X$ and $B \subseteq Y$ be subsets equipped with the corresponding subspace topology. The product topology on $A \times B$ coincides with the subspace topology on $A \times B \subseteq X \times Y$ when $X \times Y$ is given the product topology.*

Proof. When regarding $A \times B$ with the product topology, we have a basis given by

$$\mathscr{B}_{\text{prod}} = \{(U \cap A) \times (V \cap B) \mid U \text{ open in } X \text{ and } V \text{ open in } Y\}.$$

When regarding $A \times B$ under the subspace topology, combining Lemma 1.4.5 and Theorem 1.3.1 gives us the basis

$$\mathscr{B}_{\text{sub}} = \{(U \times V) \cap (A \times B) \mid U \text{ open in } X \text{ and } V \text{ open in } Y\}.$$

Basic properties of set theory tell us that

$$(U \cap A) \times (V \cap B) = (U \times V) \cap (A \times B).$$

We conclude that the two bases are actually equal, and thus the two topologies coincide. $\square$

1.5 Closed sets

Definition 1.5.1. Let $(X, \mathcal{T}_X)$ be a topological space. A subset $C \subseteq X$ is **closed** if its complement $X \setminus C$ is open.

Remark 1.5.2. Unlike doors, subsets need not be either open or closed. For instance, in $\mathbb{R}$ with the standard topology, the subset $[0, 1)$ is neither open nor closed. Conversely, a set can be both open and closed at the same time; such a set is called **clopen**. In $X = \{a, b\}$ with the discrete topology, $\{a\}$ is clopen.

Using basic set theory, we can take the definition of open sets to given the following equivalent characterization of closed sets:

Exercise 1.5.3. Let $(X, \mathcal{T}_X)$ be a topological space. Show that the following axioms characterize closed sets:

- The empty set is closed,
- The entire space X is closed.
- The *union* of two closed sets is closed, and
- The arbitrary *intersection* of closed sets is closed.

Remark 1.5.4. In fact, the axioms of a topology could just as well have been given in terms of closed subsets: a topology on X is given by a collection of subsets $\mathcal{C}$, known as the closed subsets of X, satisfying the four axioms in Exercise 1.5.3. We would then define open open sets as those whose complements are closed.

Example 1.5.5. In $\mathbb{R}$ with the standard topology, closed intervals are closed. These are any sets of the form $[a, b]$ with $a < b$, since

$$[a, b] = \mathbb{R} \setminus ((-\infty, a) \cup (b, \infty)).$$

In fact, note that any singletons $\{a\}$ are also closed: given any real number $x \neq a$, we can find a small enough open neighborhood U_x of x such that $a \notin U_x$, and thus

$$\bigcup_{x \neq a} U_x = \mathbb{R} \setminus \{a\}$$

is open. If we want to make this precise, we can take $\varepsilon > 0$ with $\varepsilon < |x - a|$, and take $U_x = B_\varepsilon(x)$ to be the ball of radius ε centered at x.

More generally, finite unions of closed intervals are closed, and arbitrary intersections of closed intervals are closed. In particular, any finite set in $\mathbb{R}$ is a closed set, since it is the union of finitely many singletons, which are closed.

Example 1.5.6. Let X be any set equipped with the finite complement topology. Then the closed subsets are the finite subsets.

Example 1.5.7 (the Cantor set)**.** Consider following sequence of subsets of $\mathbb{R}$:

$$\begin{aligned}
C_0 &= [0,1] \\
C_1 &= [0,1] \setminus (1/3,2/3) = [0,1/3] \cup [2/3,1] \\
C_2 &= C_1 \setminus (1/9,2/9) \cup (7/9,8/9) = [0,1/9] \cup [2/9,1/3] \cup [2/3,7/9] \cup [8/9,1]
\end{aligned}$$

and so on, so that for each $n \geqslant 1$

$$C_n = \frac{C_{n-1}}{3} \cup \left(\frac{2}{3} + \frac{C_{n-1}}{3}\right)$$

where scaling and adding sets of real numbers is defined in the obvious way (by adding or scaling elements). Note that each C_n is closed in $\mathbb{R}$ for the standard topology, since it is a finite union of closed intervals. It follows that the **Cantor set**

$$C := \bigcap_{n \geqslant 0} C_n$$

is also closed. One way to describe it is that it consists of those real numbers that admit a base three expansion has the form

$$0.i_1 i_2 i_3 \cdots {}_3 \quad \text{with } i_j \in \{0,2\} \quad \text{for all } j.$$

For example, $1/3 = 0.1_3$ is in the Cantor set since $1/3 = 0.0\overline{2}_3$.

Remark 1.5.8. Beware that, as with open subsets, the notion of closed can be a little confusing when talking about subspaces. If Y is a closed subset of a topological space X and we endow Y with the subspace topology, then the closed subsets of Y are those subsets of Y that are closed in X. However, the converse does not hold. For instance, consider $\mathbb{R}$ with the standard topology, and endow $X = (0,1)$ with the subspace topology. Then $(0,1/2]$ is closed in X but not in $\mathbb{R}$.

Example 1.5.9. Consider $\mathbb{R}_{ll}$, the real numbers with the lower limit topology. The set $(-\infty,0)$ is closed, since its complement

$$\mathbb{R} \setminus (-\infty,0) = [0,\infty) = \bigcup_{r>0} [0,r)$$

is a union of open sets, and thus open.

Definition 1.5.10. Let $(X, \mathcal{T}_X)$ be a topological space and let $S \subseteq X$ be any subset. The **closure** of S (in X), often written as $\overline{S}$, is the intersection of all closed subsets of X that contain S:

$$\overline{S} := \bigcap_{\substack{S \subseteq C \\ C \text{ is closed in } X}} C.$$

Remark 1.5.11. Since an arbitrary intersection of closed subsets is closed, the closure $\overline{S}$ of any subset S is closed. In fact, $\overline{S}$ is the unique smallest closed subset of X that contains S: any other closed subset of X that contains S must contain $\overline{S}$.

Definition 1.5.12. Let $(X, \mathcal{T}_X)$ be a topological space and let $S \subseteq X$ be any subset. The **interior** of S, written S^{int}, is the union of all open subsets of X contained in S:

$$S^{\text{int}} := \bigcup_{\substack{U \subseteq S \\ U \text{ is open in } X}} U.$$

Remark 1.5.13. Since an arbitrary union of open subsets is open, the interior S^{int} of any subset S is open. In fact, it is the unique largest open set contained in S: any other open set contained in S is also contained in S^{int}.

Theorem 1.5.14. *Let $(X, \mathcal{T}_X)$ be a topological space and let $S \subseteq X$ be any subset. Fix a point x in X.*

a) *We have $x \in S^{int}$ if and only if there exists an open neighborhood of x in X that is contained in S.*

b) *We have $x \in \overline{S}$ if and only if every open neighborhood of x in X intersects S nontrivially.*

Proof. By definition,

$$S^{\text{int}} := \bigcup_{\substack{U \subseteq S \\ U \text{ is open in } X}} U.$$

An element x is in this union if and only if x is one of the sets in the union, meaning that $x \in U$ for some open set $U \subseteq S$.

To show the second statement, we prove the contrapositive of each implication.

($\Rightarrow$) Suppose there is an open neighborhood U of x with $U \cap S = \emptyset$. Then $S \subseteq X \setminus U$, and $X \setminus U$ is closed by definition. By definition of closure,

$$\overline{S} \subseteq X \setminus U.$$

On the other hand, $x \notin X \setminus U$, so $x \notin \overline{S}$.

($\Leftarrow$) Let

$$x \notin \overline{S} = \bigcap_{\substack{S \subseteq C \\ C \text{ is closed in } X}} C.$$

Thus $x \notin C$ for some closed subset C of X with $S \subseteq C$. Then $X \setminus C$ is an open neighborhood of x that does not intersect S. □

Exercise 1.5.15. Let $\mathscr{B}$ be a basis for a topology on X, and let $x \in X$.

a) Show that $x \in S^{\text{int}}$ if and only if there exists an open neighborhood of x in $\mathscr{B}$ that is contained in S.

b) We have $x \in \overline{S}$ if and only if every open neighborhood of x in $\mathscr{B}$ intersects S nontrivially.

Exercise 1.5.16. Consider the subset $S = (a, b)$ of $\mathbb{R}$.

a) Find the closure of S when $\mathbb{R}$ is given the standard topology.

b) Find the closure of S when $\mathbb{R}$ is given the lower limit topology.

c) Find the interior of S when $\mathbb{R}$ is given the standard topology.

d) Find the interior of S when $\mathbb{R}$ is given the lower limit topology.

The complement of the closure is the interior of the complement:

Lemma 1.5.17. *Let S be a subset of a topological space $(X, \mathcal{T})$. Then*

$$X \setminus \overline{S} = (X \setminus S)^{int}.$$

Proof. Fix $x \in X$. Then

$$\begin{aligned} x \in X \setminus \overline{S} &\iff x \notin \overline{S} \\ &\iff \text{there is an open neighborhood } U \text{ of } x \text{ with } U \cap S = \emptyset \quad \text{by Theorem 1.5.14} \\ &\iff \text{there is an open neighborhood } U \text{ of } x \text{ with } U \subseteq X \setminus S \\ &\iff x \in (X \setminus S)^{\text{int}} \quad \text{by Theorem 1.5.14.} \end{aligned}$$

□

Example 1.5.18. Let us find the closure of $\mathbb{Q}$ in $\mathbb{R}$ under the standard topology. Given $x \in \mathbb{R}$ and any open neighborhood U of x in $\mathbb{R}$, $(x - \varepsilon, x + \varepsilon) \subseteq U$ for some $\varepsilon > 0$. By the Density of the Rationals in the real numbers, there exists $q \in \mathbb{Q}$ with $x < q < x + \varepsilon$, and hence

$$U \cap \mathbb{Q} \neq \emptyset.$$

Therefore, $x \in \overline{\mathbb{Q}}$, and we conclude that $\overline{\mathbb{Q}} = \mathbb{R}$.

Exercise 1.5.19. Consider $X = [0, 1]$ with the subspace topology for the standard topology on $\mathbb{R}$. Let C be the Cantor set and $U = X \setminus C$. Find the closure of U in X.

Exercise 1.5.20. Find the closure of $\{\frac{1}{n} \mid n \in \mathbb{N}\}$ in $\mathbb{R}$ with the standard topology, and in $\mathbb{Q}$ with the subspace topology.

Appendix A

Sets and functions

In order to do any topology, we need to be comfortable with elementary set theory. Munkres [Mun00] includes an introduction to this topic; we include here the most important definitions, facts, and notation for completeness, but this is intended as a very quick refresher.

A.1 Sets

Here is some of the notation we will use for sets:

$a \in A$	a is an element of the set A
$a \notin A$	a is not an element of A, which is the negation of $a \in A$
$P(x)$	a statement about the variable x; may be true or false depending on x
$\{x \in D \mid P(x)\}$ also denoted by $\{x \in D \colon P(x)\}$	The set of all elements of D for which $P(x)$ is true (a subset of D)
$A \subseteq B$	A is a **subset** of B $\iff$ If $a \in A$, then $a \in B$.
$A \subsetneq B$	$A \subseteq B$ and $A \neq B$
$A \supseteq B$	B is a subset of A (we also say that A is a superset of B)
$A \supsetneq B$	$A \supseteq B$ and $A \neq B$
$A \nsubseteq B$	A is not a subset of B, which is the negation of $A \subseteq B$
$A \setminus B$	$\{a \in A \mid a \notin B\}$, the complement of A and B, read A *except* B
$A \cap B$	$\{x \mid x \in A \text{ or } x \in B\}$, the intersection of the sets A and B
$A \cup B$	$\{x \mid x \in A \text{ and } x \in B\}$, the union of the sets A and B
$\emptyset$	The **empty set** (the set with no elements), also known as the **null set**

Remark A.1.1. Note that Munkres uses the notation $A - B$ for set complement, but we will not be following that convention.

The set of real numbers is denoted by $\mathbb{R}$. The set of **integers** is a subset of real numbers, given by

$$\mathbb{Z} := \{\dots, -2, -1, 0, 1, 2, \dots\}.$$

We may use subscripts to indicate certain subsets of $\mathbb{Z}$, such as

$$\mathbb{Z}_{>0} := \{a \in \mathbb{Z} \mid a > 0\}.$$

The set of **natural numbers** is the subset of the integers given by

$$\mathbb{N} := \{0, 1, 2, \dots\}.$$

Be warned that some mathematicians actually choose to not include 0 in the natural numbers. Finally, the **rational numbers** is the set $\mathbb{Q}$ of all real numbers that can be written in the form $\frac{a}{b}$ with a and b integers and $b \neq 0$.

Here are some properties of sets:

Theorem A.1.2 (Algebra of set operations)**.** *Let A, B, and C be subsets of some universe U. Then the following hold:*

Properties of $\emptyset$ and the universe	$A \cap \emptyset = \emptyset$ $A \cup \emptyset = A$	$A \cap U = A$ $A \cup U = U$
Idempotent Laws	$A \cap A = A$	$A \cup A = A$
Commutative Laws	$A \cap B = B \cap A$	$A \cup B = B \cup A$
Associative Laws	$(A \cap B) \cap C = A \cap (B \cap C)$ $(A \cup B) \cup C = A \cup (B \cup C)$	

Theorem A.1.3 (Properties of the complement)**.** *Let A and B be subsets of some universe U. With U fixed, we write $A^c := U \setminus A$. Then the following hold:*

Basic properties	$(A^c)^c = A$ $A \setminus B = A \cap B^c$	
Empty set and universe	$A \setminus \emptyset = A$ $A \setminus U = \emptyset$	$\emptyset^c = U$ $U^C = \emptyset$
Subsets and complements	$A \subseteq B \iff B^c \subseteq A^c$	

Theorem A.1.4 (Distributive Laws)**.** *Given a set A and sets B_α index by a set Λ,*

$$A \cap \left(\bigcup_{\alpha \in \Lambda} B_\alpha \right) \quad \textit{and} \quad A \cup \left(\bigcap_{\alpha} B_\alpha \right) = \bigcap_{\alpha \in \Lambda} A \cup B_\alpha.$$

Theorem A.1.5 (De Morgan's Laws)**.** *Given a set A and sets B_α index by a set Λ,*

$$A \setminus \left(\bigcup_{\alpha \in \Lambda} B_\alpha \right) = \bigcap_{\alpha \in \Lambda} (A \setminus B_\alpha) \quad \text{and} \quad A \setminus \left(\bigcap_{\alpha} B_\alpha \right) = \bigcup_{\alpha \in \Lambda} (A \setminus B_\alpha).$$

Definition A.1.6. Given a set A, the **power set** of A is the set $\mathcal{P}(A)$ of all subsets of A.

Exercise A.1.7. Show that if A is a finite set with n elements, then $\mathcal{P}(A)$ is a finite set with 2^n elements.

Note that if A is infinite then $\mathcal{P}(A)$ is infinite.

Definition A.1.8. Let A be a set.

- We say that A is **finite** if A is empty or there exists an integer $n \geqslant 1$ and a bijection $f \colon A \longrightarrow \{1, \dots, n\}$.
- We say that A is **countably infinite** if there is a bijection $f \colon A \longrightarrow \mathbb{Z}_{>0}$.
- We say that A **countable** if it is finite or countably infinite.
- We say that A is **uncountable** if it is not countable.

Given a collection of sets, there are multiple ways we can use them to create a new set.

Definition A.1.9. Given two sets A and B, the **cartesian product**, or more simply the **product** of A and B is the set of **ordered pairs**

$$A \times B := \{(a, b) \mid a \in A, b \in B\}.$$

More generally, given a collection of sets A_α indexed by a set I, their cartesian product is the set

$$\prod_{\alpha \in I} A_\alpha = \{(a_\alpha)_{\alpha \in I} \mid a_\alpha \in A_\alpha \text{ for all } \alpha \in I\}$$

of tuples (a_α), where a tuple (a_α) corresponds to an element $a_\alpha \in A_\alpha$ for each α.[1]

One can then write the **Axiom of Choice** as a statement about Cartesian products:

If the $A_\alpha \neq \emptyset$ for every $\alpha \in I$, then

$$\prod_{\alpha \in I} A_\alpha \neq \emptyset.$$

[1]It is a bit obnoxious to define this formally. Here is the proper definition: the Cartesian product of $\{A_\alpha\}_{\alpha \in I}$ is the set of all functions

$$f \colon I \longrightarrow \bigcup_{\alpha \in I} A_\alpha$$

such that $f(\alpha) \in A_\alpha$ for all $\alpha \in I$. Each such function is an element in the Cartesian product; this corresponds precisely to picking out an element $a_\alpha \in A_\alpha$ for each α.

A.2 Functions

Definition A.2.1. A **function** from a set A to a set B is a rule that associates to each element of $x \in A$ exactly one element of the set B. We usually give functions a name, and denote the function f from A to B by $f\colon A \to B$. The function $f\colon A \to B$ has three components:

- The set A is called the **domain** of f.
- The set B is called the **codomain** of f.
- The rule that associates elements from the domain with elements of the codomain.

In particular, note that while we may refer to the function as simply "the function f", indicating a rule without indicating the domain and codomain does not define a function.

Definition A.2.2. Let $f\colon A \to B$ be a function with domain A and codomain B. Given $a \in A$, we write $f(a)$ for the image of a under f, meaning the element of B that f associates to a. If $f(a) = b$, then a is called a **preimage** of b. Given a subset $C \subseteq A$, the **image** of C under f is

$$f(C) := \{f(c) \mid c \in C\}.$$

The **image** of f is the set

$$f(A) := \{f(a) \mid a \in A\}.$$

The image of f is also sometimes denoted by $\operatorname{im}(f)$ or $\operatorname{image}(f)$, and it is sometimes instead called the **range** of f.

Given a subset $C \subseteq B$, the **preimage** of C under f is

$$f^{-1}(C) = \{a \in A \mid f(a) \in C\}.$$

Remark A.2.3. Given a function $f\colon A \longrightarrow B$, note that the image of f is a subset of the codomain B of f. Moreover, the preimage of any subset C of B is a subset of the domain A.

Definition A.2.4. Given a set A, the **identity function** on A is the function $\mathrm{id}_A\colon A \longrightarrow A$ given by $\mathrm{id}_A(a) = a$ for all $a \in A$. Given sets $A \subseteq B$, the **inclusion** of A into B is the function $\iota\colon A \longrightarrow B$ given by $\iota(a) = a$ for all $a \in A$.

Definition A.2.5. Let $f\colon A \longrightarrow B$ be a function.

- We say f is **injective** if
  $$f(x) = f(y) \implies x = y.$$
  Equivalently, $|f^{-1}(\{b\})| \leqslant 1$ for every $b \in B$.
- We say f is **surjective** if $f(A) = B$, or equivalently, if for all $b \in B$ there exists $a \in A$ such that $f(a) = b$. Equivalently, $f^{-1}(\{b\})$ is nonempty for every $b \in B$.
- We say f is **bijective** if f is both injective and surjective.

Exercise A.2.6. Let $f\colon A \longrightarrow B$ be a function.

a) Show that f is injective if and only if there exists a function $g\colon B \longrightarrow A$ such that $gf = \mathrm{id}_A$.

b) Show that f is surjective if and only if there exists a function $g\colon B \longrightarrow A$ such that $fg = \mathrm{id}_B$.

c) Show that f is bijective if and only if f has an inverse, meaning that there exists a function $g\colon B \longrightarrow A$ such that $fg = \mathrm{id}_B$ and $gf = \mathrm{id}_A$.

Definition A.2.7. Given a collection of sets $\{A_\alpha\}_{\alpha\in\Lambda}$ indexed by a set Λ and a fixed $\beta \in \Lambda$, the **projection map** $\pi_\beta\colon \prod\limits_{\alpha\in\Lambda} A_\alpha \longrightarrow A_\beta$ is the function given by

$$\pi_\beta((a_\alpha)_{\alpha\in\Lambda}) = a_\beta.$$

Definition A.2.8. The **product function** of $f_\alpha\colon A \longrightarrow B_\alpha$ with $\alpha \in I$ is the function

$$(f_\alpha)_{\alpha\in I} : A \longrightarrow \prod_{\alpha\in I} B_\alpha$$

given by

$$(f_\alpha)_{\alpha\in I}(a) = (f_\alpha(a))_{\alpha\in I}.$$

Lemma A.2.9. *Consider sets A and B_α for each $\alpha \in I$.*

a) For every function $f\colon A \longrightarrow B_\alpha$ there exist functions $f_\alpha\colon A \longrightarrow B_\alpha$ such that

$$f = (f_\alpha)_{\alpha\in I} \quad \textit{and} \quad \pi_\beta \circ f = f_\beta \textit{ for all } \beta \in I.$$

b) Given functions $f_\alpha\colon A \longrightarrow B_\alpha$, the product function $f = (f_\alpha)_{\alpha\in I}$ satisfies

$$\pi_\beta \circ f = f_\beta \textit{ for all } \beta \in I.$$

A.3 Equivalence relations and order relations

Definition A.3.1. A **relation** on a set A is a subset of the cartesian product $A \times A$. We sometimes denote the relation by a symbol, such as $\sim$, and write $a \sim b$ to indicate that the pair (a, b) is an element of the relation. We may also write $a \nsim b$ to indicate that (a, b) is not in the relation.

Definition A.3.2. An **equivalence relation** on a set X is a relation $\sim$ satisfying the following properties:

- Reflexivity: $x \sim x$ for all $x \in X$.
- Symmetry: if $x \sim y$ then $y \sim x$.
- Transitivity: if $x \sim y$ and $y \sim z$, then $x \sim z$.

Given an equivalence relation $\sim$ on X, the **equivalence class** of $x \in X$ is the set

$$[x] := \{y \mid y \sim x\}.$$

An equivalence class is any subset of X of the form $[x]$ for some $x \in X$.

Remark A.3.3. Any equivalence relation on a set X partitions X by equivalence classes: every element of X is in some equivalence class, and any two distinct equivalence classes are disjoint. Thus we can write X as a disjoint union of equivalence classes:

$$X = \bigcup_{x \in X} [x].$$

We will often think of the resulting set of equivalence classes. Informally, one may think about starting from a set X and *glueing* elements together to make each equivalence class, forming a new (potentially smaller) set.

Index

Nomenclature

$\prod_{\alpha \in I} A_\alpha$	cartesian product of the sets A_α, where $\alpha \in I$
$\mathbb{R}_\ell$	$\mathbb{R}$ with the lower limit topology
$\mathbb{R}_K$	$\mathbb{R}$ with K-topology

Bibliography

[Hat02] Allen Hatcher. *Algebraic Topology.* Cambridge University Press, Cambridge, 2002.

[Mun00] James R. Munkres. *Topology.* Prentice Hall, Inc., 2 edition, 2000.