Symbolic powers
Algebra (Day 1)

BRIDGES Day 1
Eloísa Grifo

theorem (Fundamental theorem of Arithmetic)
Every positive integer n can be written as a product

$$n = p_1^{a_1} \cdots p_k^{a_k}$$

of primes p_i (with $a_i \geq 1$) which is unique up to the order of the factors

How about in other rings?

For us, rings are always commutative with 1.

Example $R = \mathbb{Z}[\sqrt{-5}] \cong \mathbb{Z}[x]/(x^2+5)$

$$6 = 2 \cdot 3 = (1+\sqrt{-5})(1-\sqrt{-5})$$

these are distinct products in irreducibles.

What's wrong?
We are focusing on elements when we should be focusing on ideals!

Def An ideal I is prime if $xy \in I \Rightarrow x \in I$ or $y \in I$

Def: An ideal I is primary if

$$xy \in I \Rightarrow x \in I \text{ or } y^n \in I \text{ for some } n$$

Def the radical of I is $\sqrt{I} := \{f \in R \mid f^n \in I \text{ for some } n\}$

Def An ideal I is primary if

$$xy \in I \implies x \in I \text{ or } y \in \sqrt{I}$$

Notes:
- P prime $\implies P = \sqrt{P}$
- I primary $\implies \sqrt{I}$ prime

Def Q is P-primary if Q is primary and $\sqrt{Q} = P$.

Warning $\sqrt{I}$ prime $\not\Rightarrow$ I primary

Exercises

1. Q_1, Q_2 P-primary $\implies Q_1 \cap Q_2$ P-primary
2. $\sqrt{I}$ maximal $\implies I$ primary

Def A primary decomposition of I is a collection of primary ideals Q_i

$$I = Q_1 \cap \cdots \cap Q_k$$

(each Q_i is called a (primary) component of I)

A primary decomposition is irredundant if
- no Q_i can be deleted
- $\sqrt{Q_i} \neq \sqrt{Q_j}$ for $i \neq j$

Note Any primary decomposition can be made irredundant by
- deleting unnecessary components
- intersecting components with the same radical

<u>Theorem</u> (Lasker, 1905, Noether, 1921)

Every ideal in a Noetherian ring has a primary decomposition

* <u>Noetherian rings</u> = Commutative algebraists favorite rings

R is Noetherian if every ideal in R is finitely generated

(big example: $R = k[x_1, \ldots, x_d]/I$ k field)

<u>Examples</u>

a) <u>Fact</u>: I any ideal

$$\sqrt{I} = \bigcap_{\substack{Q \supseteq I \\ Q \text{ prime}}} Q = \bigcap_{\substack{Q \supseteq I \\ \text{minimal} \\ \text{wrt} \supseteq I}} Q$$

Q is a <u>minimal prime</u> of I if $Q \supseteq \underset{\text{prime}}{P} \supseteq I \Rightarrow Q = P$

$\mathrm{Min}(I) := \{ Q \text{ minimal prime of } I \}$

<u>Fact</u>: Min(I) is a finite set (in any Noetherian ring).

so if $I = \sqrt{I}$ (I is a <u>radical ideal</u>), then

$$I = \underbrace{P_1 \cap \cdots \cap P_k}_{\text{primary components of } I} \qquad P_i \text{ minimal primes}$$

b) $I = (xy, xz, yz) = (x,y) \cap (y,z) \cap (x,z) \subseteq k[x,y,z]$

c) Primary decompositions are not unique $\ddot{\frown}$

$$I = (x^2, xy) \subseteq k[x,y]$$

$$= (x) \cap (x^2, xy, y^2) = (x) \cap (x^2, xy, y^n)$$

these are <u>different</u> primary decompositions. But what do they have in common?

$$(x) \cap (x^2, xy, y^2) = (x) \cap (x^2, xy, y^n)$$

$$\Big\downarrow \text{ radical}$$

$$(x), \ (x,y) \qquad\qquad (x), \ (x,y)$$

<u>theorem</u> (Uniqueness of primary decompositions)

R Noetherian ring
I ideal in R

$I = Q_1 \cap \cdots \cap Q_k$ irredundant primary decomposition

1. $\{\sqrt{Q_1}, \dots, \sqrt{Q_k}\}$ is unique and does not depend on our choice of primary decomposition

In fact, $\{\sqrt{Q_1}, \dots, \sqrt{Q_k}\} = \mathrm{Ass}(R/I) :=$ associated primes of I

where a prime P is <u>associated</u> to I if

$P = \{f \in R \mid fa \in I\}$ for some $a \in R$

(2) the components Q_i coming from minimal primes of I are unique and given by the following formula:

$\mathfrak{P} \in \mathrm{Min}(I) \subseteq \mathrm{Ass}(R/I) \Rightarrow$ the $\mathfrak{P}$-primary component of I is

$$IR_{\mathfrak{P}} \cap R := \{ f \in R \mid sf \in I,\ s \notin \mathfrak{P} \}$$

$$\left(\begin{aligned} &= \{ f \in R \mid \tfrac{s}{s} \cdot \tfrac{f}{1} \in IR_{\mathfrak{P}} \} \\ &= \{ f \in R \mid \tfrac{f}{1} \in IR_{\mathfrak{P}} \} \end{aligned} \right)$$

Most important facts about associated primes and primary decompositions:

- → Every (proper) ideal has associated primes
- → $\mathrm{Min}(I) \subseteq \mathrm{Ass}(R/I)$
- → associated primes that are not minimal are called embedded
- → primary decompositions are computationally difficult to find

What does it really mean to be an associated prime?

when M is an R-module (abelian group with a scalar product by elements in R)

$$\underset{\substack{\cap \\ M}}{\mathrm{ann}(m)} := \{ r \in R \mid rm = 0 \} \quad \text{annihilator of } m$$

$P \in \mathrm{Ass}(R/I) \iff P = \mathrm{ann}(a+I)$ for some $a \in R$

$\iff$ there exists an inclusion $R/P \hookrightarrow R/I$

Symbolic powers

Let I be an ideal in R

$I^n := (f_1 \cdots f_n \mid f_i \in I)$ nth power of I

Example $(x,y)^2 = (x^2, xy, y^2)$

P prime ideal $\rightsquigarrow$ P^n is not necessarily primary!

Example $R = \dfrac{k[x,y,z]}{(xy-z^2)}$ $P = (x,z)$ (in R)

P^2 is not primary! Because

$xy = z^2 \in P^2$, but $x \notin P^2$, $y \notin \sqrt{P^2} = P$

but P^n has a primary decomposition

What primes are going to appear?

$\mathrm{Min}(P^n) = \{P\}$ since $P^n \subseteq Q \overset{Q \text{ prime}}{\Longrightarrow} P \subseteq Q$

so an irredundant primary decomposition of P^n looks like

$$P^n = \underbrace{Q_P}_{\substack{\uparrow \\ P\text{-primary} \\ \text{component}}} \cap \underbrace{Q_2 \cap \cdots \cap Q_k}_{\substack{\text{embedded} \\ \text{components}}}$$

and we know $Q_P = \{ f \in R \mid sf \in P^n \text{ for some } s \notin P \}$

<u>Definition</u> P prime ideal
the n-th symbolic power of P is given by

$$P^{(n)} := \{ f \in R \mid sf \in P^n \text{ for some } s \notin P$$

$= P$-primary component of P^n

$=$ smallest P-primary ideal containg P

exercise

<u>Note</u>: $P^n \subseteq P^{(n)}$ always! But in general, $P^n \subsetneq P^{(n)}$

<u>Example</u> $R = \dfrac{k[x,y,z]}{(xy - z^2)}$ $\qquad P = (x, z)$ (in R)

$xy = z^2 \in P^2$
$y \notin P$ $\implies x \in P^{(2)}$ $\qquad$ but $x \notin P^2$ (!)

so $P^2 \subsetneq P^{(2)}$

General definition $I = \sqrt{I} = P_1 \cap \cdots \cap P_k$

$$I^{(n)} = P_1^{(n)} \cap \cdots \cap P_k^{(n)}$$

Fun fact: this is actually the same we would get if we took the minimal components in an irredundant primary decomposition of I^n