Symbolic Powers
Geometry (Day 2)

Bridges 2021

Challenge What are the polynomials in $\mathbb{C}[x, y, z]$ that vanish at every point on the curve

$$C = \{(t^3, t^4, t^5) \mid t \in \mathbb{C}\} \ ?$$

Solution All the polynomials in the ideal

$$I = (\underbrace{x^3 - yz}_{f}, \underbrace{y^2 - xz}_{g}, \underbrace{z^2 - x^2y}_{h})$$

these can be calculated explicitly as

$$\ker \left(\begin{array}{ccc} \mathbb{C}[x,y,z] & \longrightarrow & \mathbb{C}[t] \\ x & \longmapsto & t^3 \\ y & \longmapsto & t^4 \\ z & \longmapsto & t^5 \end{array} \right)$$

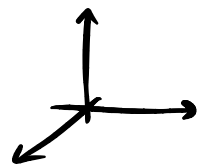
$\rightsquigarrow$ A computer can calculate these! (try Macaulay2)

there is a correspondence between

nice subsets of $\mathbb{A}^d_k$ (varieties) and $\underbrace{\text{ideals}}_{\text{systems of "polynomial equations"}}$ in $k[x_1, \dots, x_d]$

Algebra $\longleftrightarrow$ Geometry

ideals $\longleftrightarrow$ varieties

$(xy, xz, yz) \longleftrightarrow$  $= \begin{cases} xy=0 \\ yz=0 \\ xz=0 \end{cases}$

$R = \mathbb{C}[x_1, \dots, x_d] \longleftrightarrow \mathbb{A}^d_{\mathbb{C}} \ (\mathbb{C}^d)$

$\underbrace{\phantom{R = \mathbb{C}[x_1, \dots, x_d]}}$
polynomials in $x_1, \dots, x_d$ with coefficents in $\mathbb{C}$

$\mathbb{A}^d_k = \{(a_1, \dots, a_d) : a_i \in k\}$

ideal $I \xmapsto{V} V(I) = \{a \in \mathbb{A}^d \mid f(a) = 0\}$

Variety = any set of points obtained as V(some ideal)

= solution set to some system of polynomial equations

$I(X) := \{f \in R \mid f(a) = 0 \text{ for all } a \in X\} \xleftarrow{I} X$ variety

Warning! this is not a bijection exactly

eg $I = (x^2) \rightsquigarrow V(I) = \{0\} \rightsquigarrow I(\{0\}) = (x)$

Note: $\sqrt{(x^2)} = (x)$

the problem is that if $f^n(a) = 0$, then $f(a) = 0$

there is a byection

$$\{\text{radical ideals}\} \underset{V}{\overset{I}{\rightleftarrows}} \{\text{varieties}\}$$

$$(x_1 - a_1, \dots, x_d - a_d) \longleftrightarrow \bullet = \{(a_1, \dots, a_d)\}$$

maximal ideals $\longleftrightarrow$ points

$$R = (1) \longleftrightarrow \emptyset$$

$$(0) \longleftrightarrow \mathbb{A}^d_{\mathbb{C}}$$

bigger ideals $\longleftrightarrow$ smaller varieties

smaller ideals $\longleftrightarrow$ bigger varieties

$$\cap \longleftrightarrow \cup$$

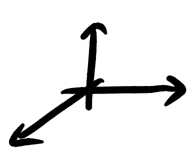
$$+ \longleftrightarrow \cap$$

<u>Exercise</u> $V(I_1 \cap I_2) = V(I_1) \cup V(I_2)$, $I(X_1) + I(X_2) = I(X_1 \cap X_2)$

prime ideals $\longleftrightarrow$ irreducible varieties (not the union of 2 smaller varieties)

$$\underbrace{I = P_1 \cap \dots \cap P_k}_{\text{primes}} \longleftrightarrow V(I) = V(P_1) \cup \dots \cup V(P_k)$$

<u>Example</u>



$$= \uparrow \cup \swarrow \cup \rightarrow$$

$$\updownarrow$$

$$(xy, yz, xz) = (x, y) \cap (y, z) \cap (x, z)$$

So given a variety X,

$$I(X) = \text{polynomials that vanish along } I$$

<u>theorem</u> (Hilbert's Nullstellensatz)

$$I = \sqrt{I} \subseteq R = \mathbb{C}[x_1, \dots, x_d]$$

then $I = \bigcap_{(a_1,\dots,a_d)\in X} (x_1 - a_1, \dots, x_d - a_d) = \bigcap_{\substack{m \supseteq I \\ m \text{ max}}} m$

<u>Challenge</u> Find the polynomials in $\mathbb{C}[x, y, z]$ that vanish <u>to order n</u> along

$$C = \{ (t^3, t^4, t^5) \mid t \in \mathbb{C} \}.$$

$\rightsquigarrow$ Wait, what does that mean?

<u>theorem</u> (Zariski-Nagata)

$$I = \sqrt{I} \subseteq \mathbb{C}[x_1, \dots, x_d]$$

$I^{(n)} = \bigcap_{\substack{m \supseteq I \\ m \text{ max}}} m^n =$ polynomials that vanish to order n at each point in $V(I)$

<u>Note</u>: $I = P_1 \cap \dots \cap P_k \rightsquigarrow I^{(n)} = P_1^{(n)} \cap \dots \cap P_k^{(n)}$

polynomials that vanish to order n at each irreducible component

$P^{(n)} = \{ f \in R \mid sf \in P^n \text{ for some } s \notin P \} =$ vanishing to order n <u>locally</u> at P

<u>Challenge</u> Find the polynomials in $\mathbb{C}[x,y,z]$ that vanish <u>to order 2</u> along

$$C = \{(t^3, t^4, t^5) \mid t \in \mathbb{C}\}.$$

$$I = (\underbrace{x^3 - yz}_{\substack{f \\ \deg 9}}, \underbrace{y^2 - xz}_{\substack{g \\ \deg 8}}, \underbrace{z^2 - x^2 y}_{\substack{h \\ \deg 10}})$$

Set:
$\deg x = 3$
$\deg y = 4$
$\deg z = 5$

<u>Answer</u>: $I^{(2)} \supsetneq I^2$

because

$$\underbrace{x}_{\deg 3} \underbrace{q}_{\deg 15} = \underbrace{fg - h^2}_{\deg 18} \in I^2, \quad x \notin I \Rightarrow q \in I^{(2)}$$

$\Rightarrow q \notin I^2$, since every element in I^2 has degree ≥ 16

<u>So</u>: $I^{(n)}$ = Elements that vanish to order n along $V(I)$

Elementary facts about symbolic powers:

① $I = I^{(1)}$

② $I^n \subseteq I^{(n)}$

③ $I^{(a)} I^{(b)} \subseteq I^{(a+b)}$

④ $I^{(n+1)} \subseteq I^{(n)}$

⑤ If I is generated by some variables, then $I^{(n)} = I^n$ for all $n \geq 1$

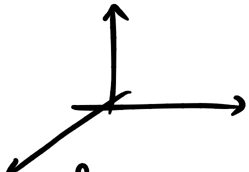
In fact, this holds more generally whenever I is a <u>complete intersection</u>

An ideal $I \subseteq \mathbb{C}[x_1, \dots, x_d]$ is a <u>complete intersection</u> if

$$\underbrace{\operatorname{codim}(V(I))}_{\underbrace{d-\dim(V(I))}_{\overset{*}{=}\ \operatorname{height}(I)}} = \underbrace{\text{minimal number of generators for } I}_{= \ \mu(I)}$$

Example $\quad I = (xy, xz, yz) \quad$ has $\mu(I) = 3$

but



has dimension $1 \equiv$ codimension $3-1=2$

so $2 < 3$

$\Rightarrow$ not CI

and actually,

$$I^{(2)} = (x,y)^{(2)} \cap (x,z)^{(2)} \cap (y,z)^{(2)}$$

$$= (x,y)^2 \cap \underset{\underset{xyz}{\cup}}{(x,z)^2} \cap (y,z)^2$$

but $xyz \notin I^2$ because every element in I^2 has degree ≥ 4.