symbolic powers

Bridges 2021

We know nothin' about Symbolic Powers (Day 3)

<u>theorem</u>

We don't know (almost) anything about symbolic powers.

Here are some big open problems about symbolic powers:

I. <u>Equality Problem</u> When is $I^n = I^{(n)}$?

① Given I, for which n do we have $I^n = I^{(n)}$?
$\longrightarrow$ not a reasonable question in general

② Fix R (eg $k[x_1, \ldots, x_d]$).
Which ideals I satisfy $I^n = I^{(n)}$ for all n?

there is a theorem of Hochster from 1964 giving necessary <u>and</u> sufficient conditions on I, but it is not pratical.

<u>Monomial ideals</u> I is a squarefree monomial ideal if it is generated by monomials of the form $x_{i_1} \cdots x_{i_n}$, where $i_k \neq i_j$ for $j \neq k$.

A monomial ideal I is <u>packed</u> if whenever we

- → set some variables $= 0$,
- → set some variables $= 1$,
- → do nothing to some variables

the resulting ideal J has codimension c, and contains c many monomials with no common variables ($\equiv$ a regular sequence of c monomials)

<u>Ex</u>: $I = (xy, xz, yz)$ has codimension 2
but any 2 monomials have a common variable $\Rightarrow$ not packed

<u>Conjecture</u> (<u>Packing Problem</u>)
Let I be a monomial ideal in $k[x_1, \ldots, x_d]$.
I satisfies $I^n = I^{(n)}$ for all $n \geq 1$ if and only if I is <u>packed</u>

③ Is it sufficient to check $I^n = I^{(n)}$ for finitely many values of n?

<u>Theorem</u> (Montaño — Núñez Betancourt, 2018)
I squarefree monomial ideal generated by μ elements
If $I^{(n)} = I^n$ for $n \leq \lceil \frac{\mu}{2} \rceil$, then $I^{(n)} = I^n$ for all $n \geq 1$.

II. Finite Generation of Symbolic Rees Algebras

$I^{(a)} I^{(b)} \subseteq I^{(a+b)}$ for all a, b

$\Longrightarrow$ can form a graded algebra

$$\mathcal{R}_S(I) := \bigoplus_{n \geq 0} I^{(n)} t^n \subseteq R[t]$$ the symbolic Rees algebra of I

Problem: Is $\mathcal{R}_S(I)$ always finitely generated?
Equivalently, is there d such that for all n,

$$I^{(n)} = \sum_{a_1 + 2a_2 + \cdots + d a_d = n} I^{a_1} \left(I^{(2)}\right)^{a_2} \cdots \left(I^{(d)}\right)^{a_d}$$

Answer: No! $\mathcal{R}_S(I)$ can be finitely generated or not.
Deciding what's the case is very hard!

Given a, b, c, let P be the defining ideal of (t^a, t^b, t^c) in $k[x,y,z]$

Is $\mathcal{R}_S(P)$ fg?

- (Goto - Nishida - Watanabe, 1994): sometimes no.
- (Huneke, Srinivasan, Cutkosky, many others): sometimes yes

II. Degrees

When I is homogeneous, $I^{(n)}$ is also homogeneous for all $n \geq 1$

Def $\alpha(I) :=$ minimal degree of a nonzero homogeneous element in I

Question: What is $\alpha(I^{(n)})$ and how does it grow with n?

$$k[x_0, \dots, x_d] \longleftrightarrow \mathbb{P}^d$$

homogeneous $I = \sqrt{I}$ $\quad$ projective varieties

$I \neq (x_0, \dots, x_d)$

$$(a_j x_i - a_i x_j \mid i \neq j) \longleftrightarrow \{(a_0 : \dots : a_d)\}$$

So when I corresponds to $\{P_1, \dots, P_s\} \subseteq \mathbb{P}^d$

$\alpha(I^{(n)}) :=$ minimal degree of a homogeneous polynomial vanishing to order n at $P_1, \dots, P_s$

$=$ smallest degree of a hypersurface passing through $P_1, \dots, P_s$ with multiplicity n

Conjecture (Chudnovsky, 1981)

If I defines s points in $\mathbb{P}^N$, then

$$\frac{\alpha(I^{(m)})}{m} \geq \frac{\alpha(I) + N - 1}{N}$$

Theorem (Bisui – G – Hà – Nguyễn)

Chudnovsky's Conjecture holds for a $\underbrace{\text{general}}$ set of $s \geq 4^N$ points

for "most" sets of points

(the sets of s points in $\mathbb{P}^N$ are parametrized by a topological space called the Hilbert scheme of s points. The theorem holds in an open dense set of the Hilbert scheme of s points)

How does one prove theorems like this?
Studying variations of the Containment problem.

Containment Problem When is $I^{(a)} \subseteq I^b$?

Theorem (Ein–Lazarsfeld–Smith 2001, Hochster–Huneke 2002, Ma–Schwede 2018)

$R = k[x_1, \dots, x_d]$, k field or $\mathbb{Z}$ or $\mathbb{Z}_p$

$I = \sqrt{I} = P_1 \cap \dots \cap P_s$

$h := \max_i \{\operatorname{codim} P_i\}$

then $\quad I^{(hn)} \subseteq I^n \quad$ for all $n \geq 1$

$\left(\Longrightarrow \quad I^{(dn)} \subseteq I^n \quad \text{for all } n \geq 1 \right)$

Example $I = (xy, xz, yz) \Rightarrow h=2 \Rightarrow I^{(2n)} \subseteq I^{n} \Rightarrow I^{(4)} \subseteq I^{2}$

Question (Huneke, 2000) What if P is a prime with $h=2$.
Is $P^{(3)} \subseteq P^{2}$?

Theorem (G, 2020) True for P defining (t^{a}, t^{b}, t^{c}) in char $\neq 3$

Conjecture (Harbourne, 2008) $I^{(hn-h+1)} \subseteq I^{n}$ for all $n \geq 1$

Theorem (Dumnicki – Szemberg – Tutaj-Gasińska, 2013) False
$\rightsquigarrow$ Constructed 12 points in $\mathbb{P}^{2}$ that don't satisfy $I^{(3)} \subseteq I^{2}$.
Extended by Harbourne-Seceleanu, and many others

But Harbourne's Conjecture is satisfied by
- $\longrightarrow$ squarefree monomial ideals
- $\longrightarrow$ general points in $\mathbb{P}^{2}$ (Bocci – Harbourne) and $\mathbb{P}^{3}$ (Dumnicki)
- $\longrightarrow$ ideals defining F-pure rings (G-Huneke, 2019)

$\left\{\begin{array}{l} \bullet\ R/I \cong k[\text{all monomials of degree } d \text{ in } v \text{ vars}] \quad \text{Veronese} \\ \bullet\ I = t\text{-minors of a generic } n \times m \text{ matrix} \\ \bullet\ \text{nice rings of invariants} \end{array}\right.$