# Defining equations for matroid varieties

## Commutative and Homological Algebra Market Presentations (CHAMP) Virtual Seminar

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background from *Euclidea*

## Thank-you to the organizers, and for the invitation to speak!

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- Joint w/ Jessica Sidman (MHC) and Will Traves (USNA).
- Goal: To find defining equations for matroid varieties.
- Ground field: $\mathbb{C}$

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# Introduction & Problem

Q: Which collections of 3 columns form a basis for the column space of $A$?

$$A = \begin{pmatrix} -1 & 1 & 2 & 3 & 4 \\ -1 & 1 & 2 & 0 & -2 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

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Look for non-zero 3-minors of $A$.

Number the columns of $A$ 1 through 5. Here are the 3-minors, according to their column indices:

$$\begin{array}{ccc}
\{1,2,3\} \to 0 & \{1,2,4\} \to -6 & \{1,2,5\} \to -12 \\
\{1,3,4\} \to -9 & \{1,3,5\} \to -18 & \{1,4,5\} \to -9 \\
\{2,3,4\} \to 3 & \{2,3,5\} \to -6 & \{2,4,5\} \to -11 \\
\{3,4,5\} \to 0 & &
\end{array}$$

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The **matroid** $\mathcal{M}_A$ on $A$ is given by the set:

$\mathcal{B} = \{$all 3-tuples except $\{1, 2, 3\}$ and $\{3, 4, 5\}\} \subset \{1, \ldots, 5\}$

The sets in $\mathcal{B}$ are called **bases** for the matroid $\mathcal{M}_A$.

Matroids are a generalization of the collections of linearly independent columns of a matrix.

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Q: Given a matroid on a matrix $A$, what other matrices have the same matroid as $A$?

E.g., rref $A$.

In fact, any other matrix that represents the same point in the Grassmannian $\mathrm{Grass}(3, 5)$!

... What else?

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Say $A$ represents a point $x \in \mathrm{Grass}(r, n)$. Put $\mathcal{M}_x = \mathcal{M}_A$.

Gel'fand, Goresky, Macpherson, and Serganova (1987) introduced the matroid stratification of $\mathrm{Grass}(r, n)$ by sets of the form

$$\Gamma_x = \{y \in \mathrm{Grass}(r, n) \mid \mathcal{M}_y = \mathcal{M}_x\}$$

and gave beautiful connections to combinatorics.

We study the Zariski closures of these strata, $\mathcal{V}_x$. $\mathcal{V}_x$ is called the **matroid variety** for the matroid $\mathcal{M}_x$.

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Let $X$ denote the generic $r \times n$ matrix and $\wedge^r X$ the matrix whose entries are the $r$-minors of $X$.

Given a matroid variety $\mathcal{V}_x$, we wish to find the defining ideal,

$$\begin{aligned} I_x = I(\mathcal{V}_x) &\subseteq \mathbb{C}[\mathrm{Grass}(r,n)] \\ &= \frac{\mathbb{C}[X]}{\langle \text{Plücker relations} \rangle} \\ &= \mathbb{C}[\wedge^r X]. \end{aligned}$$

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Mnëv (1985), Sturmfels (1989), Knutson, Lam, & Speyer (2013): **All "hell" breaks loose when we try to find $I_x$.**

Why?

- Mnëv, Sturmfels: Arbitrary singularities appear.
- The Zariski topology; the matroid stratification is based on Zarski-open conditions, the basis axioms of a matroid.
- The equations are not obvious (as we shall see).

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If $A_x$ is a matrix representative of $x \in \mathrm{Grass}(r, n)$, $\lambda$ an ordered subset of $\{1, \dots, n\}$ of size $r$, then let $[\lambda]$ denote the $r$-minor of $A_x$ with columns indexed by $\lambda$.

Define the ideal

$$N_x = \langle [\lambda] \mid \lambda \text{ is not a basis of } \mathcal{M}_x \rangle \subseteq I_x.$$

(N for "Naïve")

- Knutson, Lam, & Speyer: $N_x = I_x$ when $\mathcal{M}_x$ is a **positroid**. (Is the converse true?)
- Sturmfels (1993): $N_x \subsetneq I_x$ in general.

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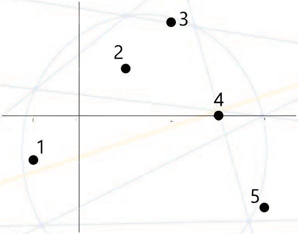
# Paradigm shift

## Matroids as point configurations

Idea: Think of the columns of a matrix $A$ as points in $\mathbb{P}^{r-1}$.




Q: What do you notice about the points in relation to the matroid $\mathcal{M}_A$ whose bases are

$$\{\text{all 3-tuples except } \{1,2,3\} \text{ and } \{3,4,5\}\} \subset \{1,\dots,5\}?$$

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Points 1, 2, and 3 are collinear if and only if $\{1, 2, 3\}$ is not a basis for $\mathcal{M}_A$. We say the **bracket** $[123]$ vanishes. We also have $[345] = 0$.

So if $x \in \mathrm{Grass}(3, 5)$ is the point represented by $A$, then we have

$$N_x = \langle [123], [345] \rangle.$$

Q: Is $N_x = I_x$? (Yes in this case, actually – $\mathcal{M}_x$ is positroid.)

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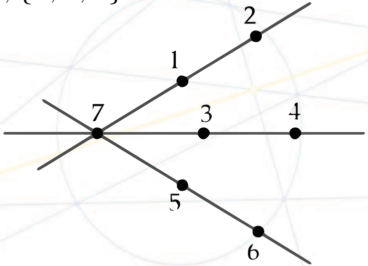
## Example (Ford 2013):

$\mathcal{M}_{\text{pencil}}$ = matroid on a $3 \times 7$ matrix with *non*bases $\{1,2,7\},\{3,4,7\},\{5,6,7\}$




$N_{\text{pencil}} = \langle [127], [347], [567] \rangle$.

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All hell breaks loose:

$I_{\text{pencil}} = N_{\text{pencil}} + \langle [134][256] - [234][156] \rangle$. How could we have known this?!

Ans: The **Grassmann-Cayley algebra**.

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# Tool

## Grassmann-Cayley algebra

The Grassmann-Cayley algebra was invented to do *synthetic projective geometry*.

We do arithmetic on the points (vectors) themselves, and obtain expressions in the brackets.

The Grassmann-Cayley algebra is the usual exterior algebra over the vector space $\mathbb{C}^r$ equipped with two operations:

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**join** ($\vee$): refers to the line passing through points (in $\mathbb{P}^2$)


The line joining 1 and 2 is $1 \vee 2$, or 12.

When $r = 3$ (i.e., over $\mathbb{P}^2$), three points joined makes a bracket, and three collinear points make the bracket vanish.

(The join is the usual exterior product with the wedge symbol flipped upside down.)

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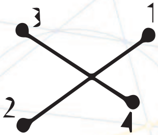
**meet** ($\wedge$): refers to the intersection of two lines (in $\mathbb{P}^2$)




The meet of the lines $1 \vee 2$ and $3 \vee 4$ is $(1 \vee 2) \wedge (3 \vee 4)$, or $12 \wedge 34$.

The meet operation uses **shuffle products**, which also produce expressions in the brackets.

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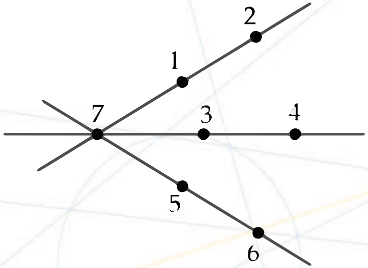
For general $r$, given two extensors $v = v_1 \cdots v_k$ and $w = w_1 \cdots w_l$,

- when $k + l < r$ we define the meet to be 0;
- if $k + l \geq r$, the meet is defined to be

$$v \wedge w = \sum_{\sigma \in \mathcal{S}(k,\ell,r)} \operatorname{sign}(\sigma)[v_{\sigma(1)} \cdots v_{\sigma(r-\ell)} w_1 \cdots w_\ell] \cdot v_{\sigma(r-\ell+1)} \cdots v_{\sigma(k)},$$

where $\mathcal{S}(k, \ell, r)$ is the set of all permutations $\sigma$ of $\{1, \ldots, k\}$ so that $\sigma(1) < \cdots < \sigma(r - \ell)$ and $\sigma(r - \ell + 1) < \cdots < \sigma(k)$.




We have $((1 \vee 2) \wedge (3 \vee 4)) \vee 5 \vee 6 = 0$, because these three points are collinear. Here's how to apply the shuffles:

$$\begin{aligned}(12 \wedge 34) \vee 56 &= ([134]2 - [234]1)56 \\ &= [134][256] - [234][156] = 0\end{aligned}$$

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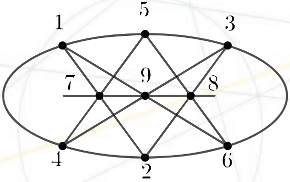
Example: **Pascal's theorem** says if six points on a conic are configured in the way illustrated below, then the resulting intersection points (7, 8, and 9) are collinear.




We have a matroid $\mathcal{M}_{\mathsf{Pascal}}$ corresponding to this configuration of points, and a matroid variety $\mathcal{V}_{\mathsf{Pascal}}$.

Q: What is $N_{\mathsf{Pascal}}$?

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# Our result and generalizations

**Theorem (Sidman, Traves, W (2020))**

*$I_{Pascal}$ is generated by at least one quartic, three independent cubics, and three independent quadrics in the brackets, besides the brackets in $N_{Pascal}$.*

Evidence (M2) says these are the only defining equations.

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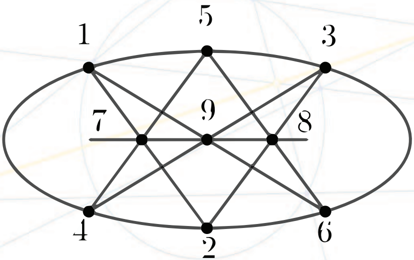
Finding the quartic: The statement of Pascal's theorem says we have $(12 \wedge 45) \vee (23 \wedge 56) \vee (34 \wedge 61) = 0$.

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Apply the shuffle products:

$$
\begin{aligned}
&(12 \wedge 45) \vee (23 \wedge 56) \vee (34 \wedge 61) \\
&= ([145]2 - [245]1) \vee ([256]3 - [356]2) \vee ([361]4 - [461]3)
\end{aligned}
$$

Now "foil":

$$
\begin{aligned}
&(([145]2 - [245]1) \vee ([256]3 - [356]2)) \vee ([361]4 - [461]3) \\
&= ([145][256]23 - [145][356]22 - [245][256]13 + [245][356]12) \\
&\qquad \vee ([361]4 - [461]3)
\end{aligned}
$$

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Finish "foiling":

$$
\begin{aligned}
&([145][256]23 - [145][356]22 - [245][256]13 + [245][356]12) \\
&\quad \vee ([361]4 - [461]3) \\
=&[145][256][361][234] - \cancel{[145][256][461][233]} \\
&\quad - \cancel{[145][356][361][224]} + \cancel{[145][356][461][223]} \\
&\quad - [245][256][361][134] + \cancel{[245][256][461][133]} \\
&\quad + [245][356][361][124] - [245][356][461][123] \\
=& [145][256][361][234] - [245][256][361][134] \\
&\quad + [245][356][361][124] - [245][356][461][123] = 0
\end{aligned}
$$

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## Why is the quartic not in $N_{\text{Pascal}}$?

Let $f$ denote the quartic. It suffices to find $y \in \mathcal{V}(N_{\text{Pascal}})$ with $f(y) \neq 0$.

Let $y \in \text{Grass}(3, 9)$ be represented by the matrix $A_y$ whose:

- first six columns correspond to six points in $\mathbb{P}^2$ not on a conic.
- last 3 columns are zero.

Recall,

$$N_{\text{Pascal}} = \langle [127], [238], [349], [457], [568], [169], [789] \rangle.$$

Every bracket in $N_{\text{Pascal}}$ involves one of the columns 7, 8, or 9, so must vanish on $y$. So $y \in \mathcal{V}(N_{\text{Pascal}})$.

On the other hand, the first six columns of $A_y$ were chosen so that the corresponding points don't satisfy Pascal's theorem, the statement encoded in $f$. So we can't have $f(y) = 0$.

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We generalize our theorem to:

- more points in the Pascal configuraion (i.e., $n \geq 6$ points on a conic).
- Pascal's theorem in higher dimensions, by a result of Caminata and Schaffler.
- configurations arising from Cayley-Bacharach theorems.

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Example: The twisted cubic.

Caminata & Schaffler (2019) give Grassmann-Cayley conditions for $d+4$ points in $\mathbb{P}^d$ to lie on a rational normal curve, $d \geq 2$.

The $d=2$ case is Pascal's theorem, with its converse (proved independently by Braikenridge and Maclaurin).

The $d=3$ case is Pascal-Braikenridge-Maclaurin's theorem for a twisted cubic.

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Let $\mathcal{P} = \{P_1, \ldots, P_7\} \subset \mathbb{P}^3$.

For each 6-tuple $\lambda = \{i_1, \ldots, i_6\} \subset \{1, \ldots, 7\}$, let $\{j\}$ denote its complement.

CS: The points in $\mathcal{P}$ lie on a twisted cubic if and only if

$$(i_1 i_2 \wedge i_4 i_5 j) \vee (i_2 i_3 \wedge i_5 i_6 j) \vee (i_3 i_4 \wedge i_6 i_1 j) \vee j = 0.$$

Each expression in parentheses is an intersection of a line and a plane, which gives a point. The statement is that the four points lie on a plane.

$$(\underbrace{12 \wedge 457}_{8}) \vee (\underbrace{23 \wedge 567}_{9}) \vee (\underbrace{34 \wedge 617}_{10}) \vee 7 = 0$$

$$(\underbrace{12 \wedge 456}_{11}) \vee (23 \wedge 576) \vee (34 \wedge 716) \vee 6 = 0$$

$$(12 \wedge 465) \vee (23 \wedge 675) \vee (\underbrace{34 \wedge 715}_{12}) \vee 5 = 0$$

$$(12 \wedge 564) \vee (\underbrace{23 \wedge 674}_{13}) \vee (\underbrace{35 \wedge 714}_{14}) \vee 4 = 0$$

$$(\underbrace{12 \wedge 563}_{15}) \vee (\underbrace{24 \wedge 673}_{16}) \vee (\underbrace{45 \wedge 713}_{17}) \vee 3 = 0$$

$$(\underbrace{13 \wedge 562}_{18}) \vee (\underbrace{34 \wedge 672}_{19}) \vee (\underbrace{45 \wedge 712}_{20}) \vee 2 = 0$$

$$(\underbrace{23 \wedge 561}_{21}) \vee (34 \wedge 671) \vee (45 \wedge 721) \vee 1 = 0$$

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Let $\mathcal{Q}$ denote the set of points labelled with the underbraces, and note that $|\mathcal{Q}| = 14$.

**Theorem (Sidman, Traves, W 2020)**

*Let the coordinates of points in $\mathcal{P}$ and $\mathcal{Q}$ comprise the columns of a $4 \times 21$ matrix $A$. Let $x \in \mathrm{Grass}(4, 21)$ denote the subspace spanned by the rows of $A$. Then there are nontrivial quartics in $I_x$.*

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## Other questions:

1. Study the matroid variety correpsonding to the configuration of points in Pappus's theorem.
2. Study the matroid varieties corresponding to graphical matroids (e.g., wheel graphs).
3. Hard: Let $J_x \subseteq \mathbb{C}[\wedge^r X]$ be the ideal generated by the product of brackets corresponding to bases of $\mathcal{M}_x$ (a "compliment" of $N_x$). It is shown (cf. STW) that the saturation of $N_x$ by $J_x$ is contained in $I_x$. In fact, $\sqrt{N_x : J_x^\infty} = I_x$. What conditions will guarantee that $N_x$ is radical? (Implies the saturation is radical)

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Thank-you!

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