Polarizations of \((x_1, \ldots, x_n)^d \subseteq k[x_1, \ldots, x_n]\)

Nyah Almousa, Cornell University

joint w/ Gunnar Fløystad + Henning Lohne

Universitet i Bergen

* Polarization?

\[ \text{maximal ideal } \mathfrak{m} \rightarrow \text{polynomial preserving basic properties} \rightarrow \text{squarefree mon. ideal} \rightarrow \text{more variables} \]

* Why?

1. Squarefree mon. ideals \(\iff\) simplicial complexes
2. Hartshorne's 1969 - connectedness of Hilbert Scheme
   \(\rightarrow\) "extensions" \(\rightarrow\) specialization of polarizations
    (strongly stable ideals)
    \(\text{build \(a\) cellular res. (of polarizations)}\)
    (nonstandard pol)
    \(\text{build \(a\) cellular res. (of polarizations)}\)
    \(\text{build \(a\) cellular res. (of polarizations)}\)
    \(\text{build \(a\) cellular res. (of polarizations)}\)

Def: \(\mathfrak{m} S = k[x_1, \ldots, x_n] \) an Artinian monomial ideal

\(d_i = \) highest power of \(x_i\) that shows up in a gen. of \(I\)
\( x_i \leftarrow \{ x_{i1}, x_{i2}, \ldots, x_{in_i} \} \)
\( S = k[x_1, \ldots, x_n] \) in variables of \( x_i \)
\( \mathcal{I} \subseteq S \) is a polarization of \( \mathcal{I} \) if
\[
\sigma = (x_{11} - x_{12}, x_{11} - x_{13}, \ldots, x_{1n_1} - x_{1d_1}) \cup
(x_{21} - x_{22}, x_{21} - x_{23}, \ldots, x_{21} - x_{2d_2}) \cup
\vdots
\cup
(x_{n_1} - x_{n_2}, \ldots, x_{n_1} - x_{nd_n})
\]
is a regular \( S/\sim \) -sequence and \( \sim \subseteq \mathcal{S} \cong \mathcal{I} \)

- **Standard Polarization**
\[
\begin{align*}
x_1 &\rightarrow x_2 & x_3 \\
x_2 &\rightarrow x_2 & x_3 \\
x_3 &\rightarrow x_1 & x_2 & x_3 & x_{32} & x_{33}
\end{align*}
\]

- **Box Polarization**
\[
\begin{align*}
x_1 &\rightarrow x_1 & x_2 & x_3 & x_{34} & x_{35} & x_{36}
\end{align*}
\]

**Visualization**

\( \Delta_n(d) := \text{lattice simplex of } a \in \mathbb{N}_0^n \text{ st. } z_0 = d \)

\( \Delta_3(2) \)

\( a \in \Delta_n(d) \leftrightarrow \text{gen of } S = k(x_1, \ldots, x_n) \)

- each edge \( \rightarrow \) linear syzygy

- What about the following?
What about the following?

\[ x_1 - x_2 \]

\[ x_1 z_2 (y_1 - y_2) = (x_1 y_1) z_2 - (y_2 z_2) x_1 = 0 \]

\[ z_1 - z_2 \]

\[ \Delta_3(3) \]

Thm (Lonne): For every choice of removing exactly one edge from each down-triangle of \( \Delta_3(3) \), if a polarization of \( (x_i, y_i, z_i) \) satisfies the corresponding cell complex supports a minimal cellular rees of the polarization, and therefore \( (x_i y_i, z_i) \).

\[ \Delta_d(3) \]

Idea: want a spanning tree of complete down-graph

\[ \Delta_n(d) : \text{ put partial orders } \preceq_i \]

where \( a \preceq_i b \) if \( a_i \leq b_i \) and \( a_j \geq b_j \) \( \forall j \neq i \)

\[ \begin{align*}
200 & \leq_1 110 \\
110 & \leq_2 200
\end{align*} \]

\[ \begin{align*}
200 & >_1 110 \\
110 & >_2 200
\end{align*} \]

\[ X_i = \{ x_{i1}, \ldots, x_{id} \} \]

\[ B(X_i) = \text{ Boolean poset on } d \text{ clans} \]

Polynomials give order preserving maps

\[ \forall \ldots \wedge (x_i) \rightarrow \text{ div} \]
Polynomials give order preserving maps

\[ X_i : \Delta_n(d) \rightarrow \mathcal{B}(X_i) \]

\[ \Delta_n[3] \]

Thm (AFL): Isotone maps \( X_1, \ldots, X_n \) determine a polynomial of \((x_1, \ldots, x_n)\) iff for every "complete down-graph" in \( \Delta_n(d) \), the linear syzygy edges contain a spanning tree.

Stanley-Reisner complex of Tots (of Artinian mon. ideals)

Thm (AFL): Let \( \Delta(I) \) be the simplicial complex associated to a poly of an Artinian monomial ideal \( I \).

Every cdim 1 face of \( \Delta(I) \) is antichain.

Björner: A constructible simplicial complex with this property is a top. ball or sphere.

Question: Are (pols) top. balls/spheres? are they constructible?

Shellable \( \Rightarrow \) constructible

\[ \Delta \]

(Shellable) \( \Rightarrow \) \( I_G \) has linear quotients

\[ I_G \text{ is sequentially CM} \]

\[ \text{Def: } I 	ext{ sq. free monomial ideal, } I^y 	ext{ is the Alexander dual of } I \text{ if the mons in } I^y \text{ are precisely those} \]

\[ f_1, \ldots, f_r \text{ s.t. } (f_1, \ldots, f_r) : f_{r+1} = (\text{vanishing}) \]
We found $\mathcal{AD}$ explicitly for $(x_1, \ldots, x_n)^2$.

**Thm (AFL):** This algorithm gives Alex Dual for pols of $(x_1, \ldots, x_n)^d$.

**Thm (AFL):** Alexander duals of $(x_1, x_2)^d$ and $(x_1, \ldots, x_n)^2$ have linear quotients.

$S = \{x_1, \ldots, x_n \}$

$\chi_i = "color\ classes"$

- A monomial is a rainbow mon. if it is div. by 1 var in each color class.

**Thm:** The Alex dual of a pol. of an Artinian mon. ideal is a rainbow mon. ideal with $\#\ color\ classes$.  

**Hilbert Schemes**

- Leave: The standard pol and the box pol of $(x_1, \ldots, x_n)$ are smooth pts on the Hilbert scheme.

**Question:** Are all polarizations of Artinian mon. ideals smooth pts on the Hilbert scheme?

- Dimensions of tangent spaces of pols of $(x, y, z)^3$...
- Dimensions of tangent spaces of poles of \((x, y, z)^5\)

**Std**: dim of tangent space is 108

**Box**