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Polarizations of $(x_1, \ldots, x_n)^d \subseteq \mathbb{k}[x_1, \ldots, x_n]$

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★ Polarization?

mon. ideal $I \xrightarrow[\text{preserving hom. properties}]{\text{polarize}} \tilde{I}$ squarefree mon. ideal

$I \subseteq S = k[x_1, \ldots, x_n]$, $\tilde{I} \subseteq \tilde{S}$ ← more variables

★ Why?

1. Squarefree mon. ideals $\xleftrightarrow[1:1]{\text{Stanley-Reisner}}$ simplicial complexes
2. Hartshorne's 1966 – connectedness of Hilbert Scheme
   → "distractions" → specialization of polarizations
3. Nagel-Reiner (2009): build a minimal cellular res. of strongly stable ideals

(strongly stable ideals) ⟶ (use a nonstandard pol) ⟶ (build a cellular res of polarization)

strongly stable ideals $\supseteq (x_1, \ldots, x_n)^d$

↓

(depolarize, same res. works)

Def: $I \subseteq S = k[x_1, \ldots, x_n]$ an Artinian monomial ideal.

- d_i = highest power of x_i that shows up in a gen of I

- $\check{X}_i = \{x_{i_1}, x_{i_2}, \dots, x_{id_i}\}$
- $\tilde{S} = k[\check{X}_1, \dots, \check{X}_n]$ in variables of $\check{X}_i$

$\tilde{I} \subseteq \tilde{S}$ is a <u>polarization</u> of I if

$$\sigma = (x_{11} - x_{12}, x_{11} - x_{13}, \dots, x_{11} - x_{1d_1}) \cup$$
$$(x_{21} - x_{22}, x_{21} - x_{23}, \dots, x_{21} - x_{2d_2}) \cup$$
$$\vdots$$
$$(x_{n1} - x_{n2}, \quad \dots \quad x_{n1} - x_{nd_n})$$

is a regular $\tilde{S}/\tilde{I}$-sequence and $\tilde{I} \otimes \tilde{S}/\sigma \cong I$

$$\underbrace{x_1^{2}}_{I} x_2 x_3^3$$

I: subset of vars in $\check{X}_i$

- Standard pol:

$$x_1^2 x_2 x_3^3 \longmapsto x_{11} x_{12} x_{21} x_{31} x_{32} x_{33}$$

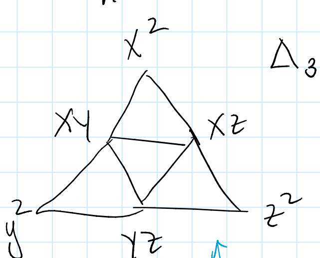
- Box polarization:

$$x_1^2 x_2 x_3^3 \longmapsto x_{11} x_{12} x_{23} x_{34} x_{35} x_{36}$$

✱ Visualization

$\Delta_n(d) :=$ lattice simplex of $\underline{a} \in \mathbb{N}_0^n$ s.t. $\sum a_i = d$

$\Delta_3(2)$

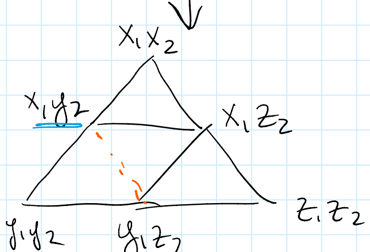


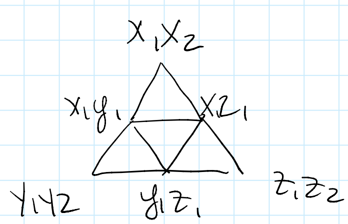
$S = k[x,y,z]$

$\underline{a} \in \Delta_n(d) \longleftrightarrow$ gen of $(x_1, \dots, x_n)^d$

each edge $\longleftrightarrow$ linear syzygy

$\downarrow$ box polarization

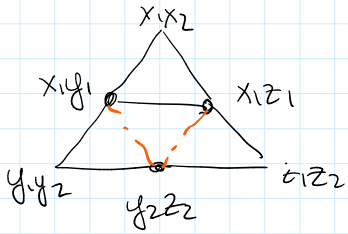




- What about the following?

x_1x_2

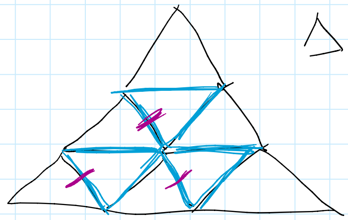
- What about the following?



$x_1 - x_2$

$$\underline{x_1} z_2 (y_1 - y_2) = \cancel{(x_1 y_1)} z_2 - \cancel{(y_2 z_2)} x_1 = 0$$

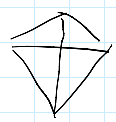
$z_1 - z_2$



$\Delta_3(3)$

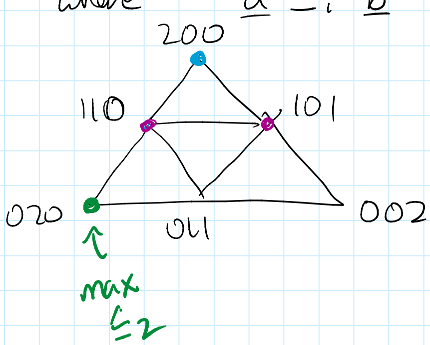
<u>Thm</u> (Lohne): For every choice of removing exactly one edge from each down-triangle of $\Delta_3(d)$, $\exists$ a polarization of $(x, y, z)^d$ s.t. the corresponding cell complex supports a min'l cellular res of the polarization (and therefore $(x, y, z)^d$)

- $\Delta_4(d)$



Idea: Want a <u>spanning tree</u> of complete down-graph

- $\Delta_n(d)$: put partial orders $\geq_i$ where $\underline{a} \leq_i \underline{b}$ if $a_i \leq b_i$ and $a_j \geq b_j$ $\forall i \neq j$



$\leq_1$ $\leq_x$

$\leq_2$

$200 >_1 110$

$110 >_2 200$

- $\check{X}_i = \{x_{i1}, \ldots, x_{id}\}$
- $B(\check{X}_i)$ = Boolean poset on d elems

Polarizations give order preserving maps

$X_i : \Delta_n(d) \longrightarrow B(\check{X}_i)$

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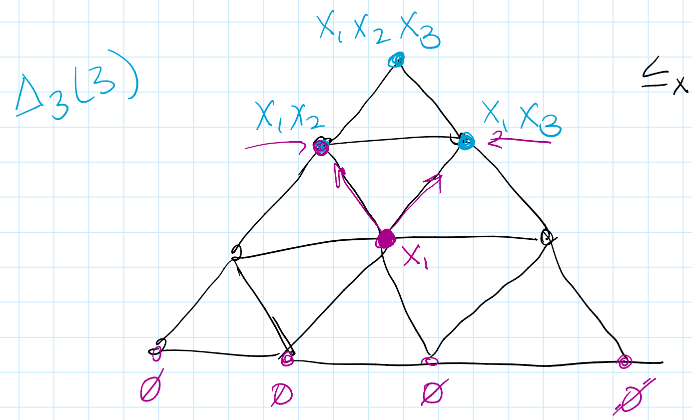


Thm (AFL): Isotone maps $X_1, \dots, X_n$ determine a polarization of $(x_1, \dots, x_n)^d$ iff for every "complete down-graph" in $\Delta_n(d)$, the linear syzygy edges contain a spanning tree.

☆ Stanley-Reisner complex of Pols (of Artinian mon. ideals)

Thm (AFL): Let $\Delta(J)$ be the simplicial complex associated to a pol. of an Artinian monomial ideal I. Then every codim 1 face of $\Delta(J)$ is contained in 1 or 2 facets.

Björner: A $\boxed{\text{constructible}}$ simplicial complex with this property is a top. ball or sphere.

Question: Are Δ(pols) top. balls / spheres?

→ are they constructible?

- Shellable ⟹ constructible

Δ (pure) shellable ⟸⟹ $\boxed{I_\Delta^\vee \text{ has linear quotients}}$

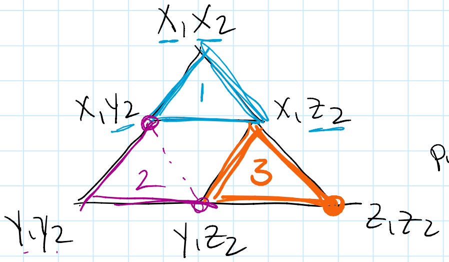
I has lin. quotients if ∃ order generators of gens of I $f_1, \dots, f_r$ s.t. $(f_1, \dots, f_k) : f_{k+1} = (\text{variables})$

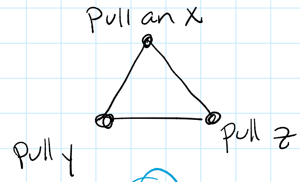
⇓ ⇓

I_Δ is ~~sequentially~~ CM ⟸⟹ $I_\Delta^\vee$ has linear free resolution

- Def: I sq. free monomial ideal. $I^\vee$ is the Alexander Dual of I if the mons in $I^\vee$ are precisely those w/ nontrivial common divisor w/ every mon. in I.

- We found AD explicitly for $(x_1, \dots, x_n)^d$



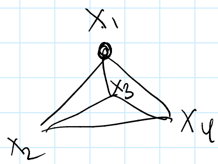


1 $x_1 y_2 z_2$, $x_2 y_2 z_2$

2 $x_1 y_1 z_2$

3 $x_1 y_1 z_1$

generates entire AD



Thm (AFL): This algorithm gives Alex. Dual for pols of $(x_1, \dots, x_n)^d$

Thm (AFL): Alexander duals of pols of $(x,y,z)^d$ and $(x_1, \dots, x_n)^2$ have linear quotients.

$\tilde{S} = k[\check{X}_1, \dots, \check{X}_n]$ $\quad$ $\check{X}_i$ = "color classes"

- A monomial is a rainbow mon. if it is div. by 1 var in each color class.

Thm: The Alex. dual of a pol. of an Artinian mon. ideal is a rainbow mon. ideal with n-linear res.
↑ # color classes.

Hilbert Schemes

- Lohne: The standard pol and the box pol of $(x_1, \dots, x_n)^d$ are smooth points on the Hilbert scheme.
- Question: Are all polarizations of Artinian mon. ideals smooth pts on the Hilbert scheme?
- Dimensions of tangent spaces of pols of $(x,y,z)^3$

- Dimensions of tangent spaces of pols of $(x, y, z)^5$

  <u>Std</u>: dim. of tangent space is 69

  <u>Box</u> 108