Characterizing Cohen-Macaulay power edge ideals of trees

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Outline

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Graph Theory

Motivation

Algebra

1. Vertex Covers

2. Edge Ideals

4. The PMU problem

5. Power Edge Ideals

6. MAIN THEOREM: Finding minimal PMU Covers

Electrical Engineering

3. Phasor Measurement Units (PMUs)

Main Part of the Talk

Vertex Covers

Definition

Let $G = (V, E)$ be a graph. A *vertex cover* of G is a subset $V' \subseteq V$ such that for each edge $v_i v_j$ in G either $v_i \in V'$ or $v_j \in V'$. A vertex cover is *minimal* if it does not properly contain another vertex cover.

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Example

For the graph $G =$

$$\begin{array}{ccc} v_1 & — & v_2 \\ & \diagup & | \\ v_4 & — & v_3 \end{array}$$

The minimal vertex covers for G are $\{v_1, v_3, v_4\}$, $\{v_2, v_3\}$, $\{v_2, v_4\}$.

Edge Ideals

Definition

Let G be a graph with vertex set $V = \{v_1, \ldots, v_d\}$. The *edge ideal* of G is the ideal $I_G \subseteq R = A[X_1, \ldots, X_d]$ that is "generated by the edges of G"

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Set $R = A[X_1, X_2, X_3, X_4]$. The graph $G =$

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We obtain the following irredundant m-irreducible decomposition:

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Vertex Covers and Edge Ideals

Theorem (R.H. Villareal)

Let G be a tree. The following conditions are equivalent:

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Example

Let

$$T = \begin{array}{ccccc} v_{1,1} & — & v_{1,2} & — & v_{1,3} \\ | & & | & & | \\ v_{2,1} & & v_{2,2} & & v_{2,3} \end{array}, \quad T' = \quad v_{1,1} — v_{1,2} — v_{1,3}$$

Note that T is the suspension of T'. Thus, T is Cohen-Macaulay.

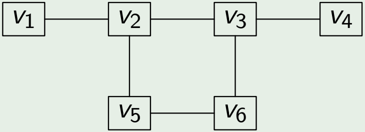
PMU Covers and Edge Ideals

Definition

- In an electrical power system, a *bus* is a substation where *transmission lines* meet.
- Each line connects two buses.

Example

The graph



represents a power system with six buses and six lines.

PMU Covers and Power Edge Ideals

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- A *Phasor measurement unit (PMU)* is a device placed at a bus in an electrical power system to monitor the voltage at the bus and the current in all lines connected to it. A *PMU placement* is a set of buses where PMUs are placed.

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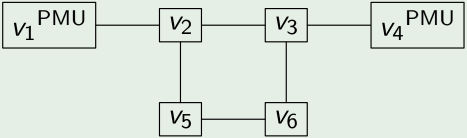

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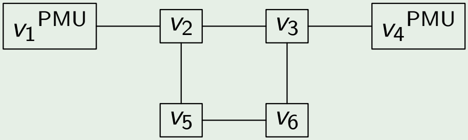

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- A bus in the system is *observable* if its voltage is known. A line is the system is *observable* if its current is known.
- A *PMU cover* of the system is a placement of some PMUs on buses making the entire system observable.

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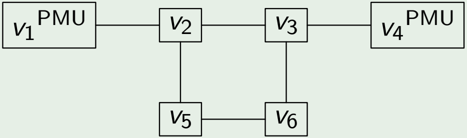

PMU Covers and Power Edge Ideals

Rule

1. *Place a PMU at vertex v. Then the vertex v is observable and every line incident to v is observable.*
2. *(Ohm) A bus incident to an observable line is observable.*
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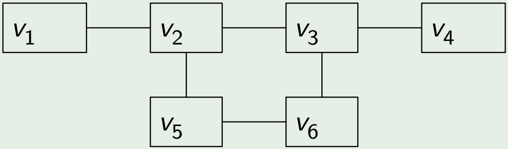

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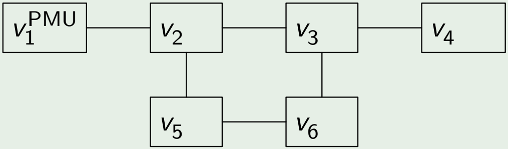

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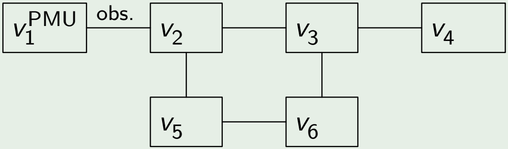

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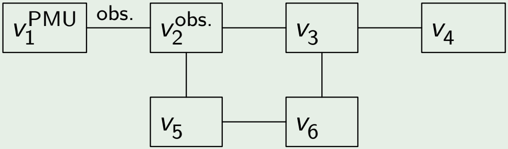

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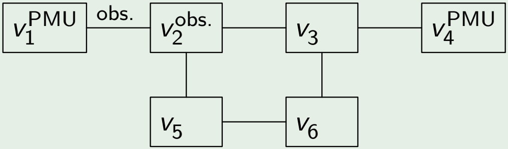

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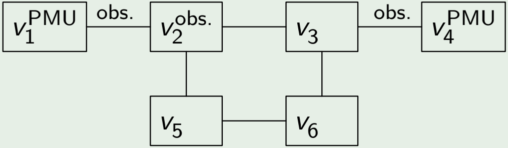

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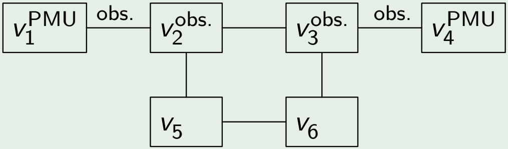

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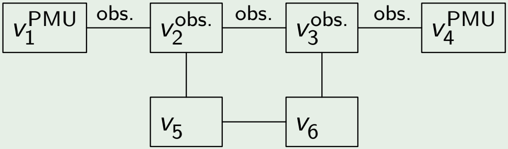

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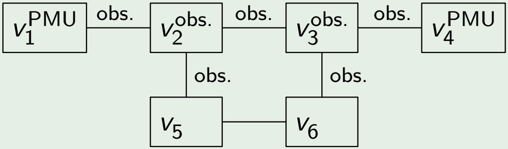

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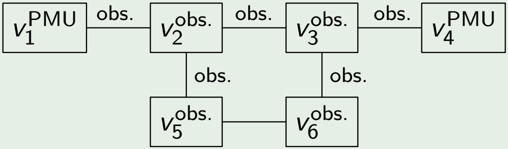

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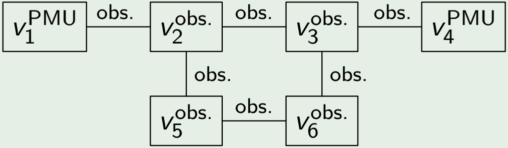

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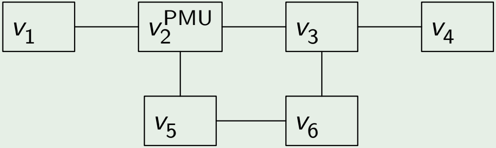

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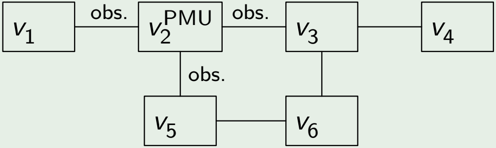

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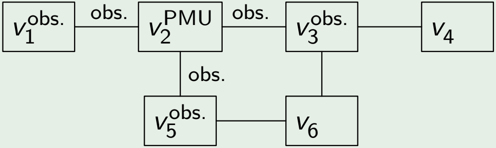

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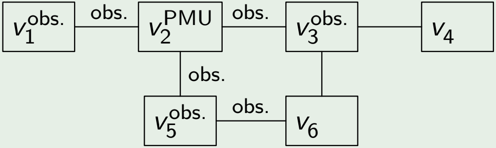

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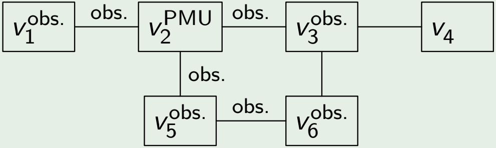

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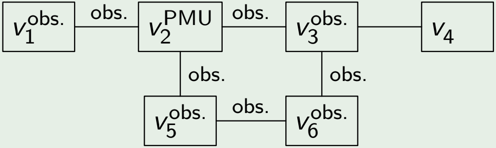

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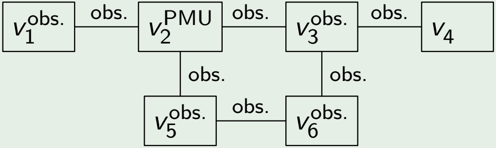

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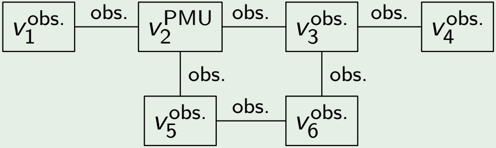

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where the intersection is taken over all PMU covers of G.

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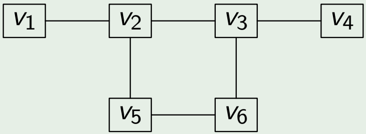

$$I_G^P = \langle X_1, X_4 \rangle \cap \langle X_2 \rangle \cap \langle X_3 \rangle \cap \langle X_5 \rangle \cap \langle X_6 \rangle = \langle X_1X_2X_3X_5X_6, X_2X_3X_4X_5X_6 \rangle$$

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Goal

Study the problem algebraically and find some interesting examples of ideals along the way. Specifically, we will:

- Characterize the trees T such that I_T^P is unmixed, i.e., such that all minimal PMU covers have the same size.
- Characterize the trees T such that I_T^P is Cohen-Macaulay.

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1. (i) G is unmixed with respect to PMU covers.
2. (ii) Every vertex of deg. ≥ 3 is adjacent to exactly 2 vertices of deg. ≤ 2.
3. (iii) G is obtained by "nicely linking" a sequence of paths.

Characterizing Unmixed Trees

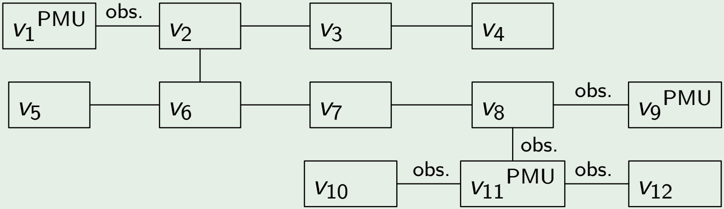
Theorem (CGHMS)

Let G be a tree. The following conditions are equivalent:

1. (i) G is unmixed with respect to PMU covers.
2. (ii) Every vertex of deg. ≥ 3 is adjacent to exactly 2 vertices of deg. ≤ 2.
3. (iii) G is obtained by "nicely linking" a sequence of paths.

   -PMU Covers: Choose one vertex from each path

Example



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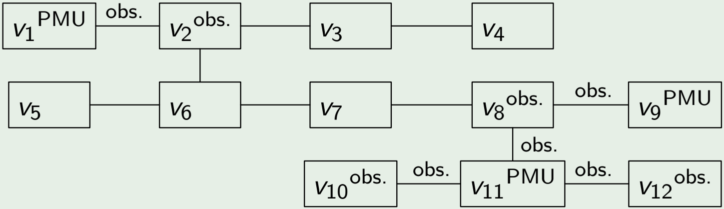

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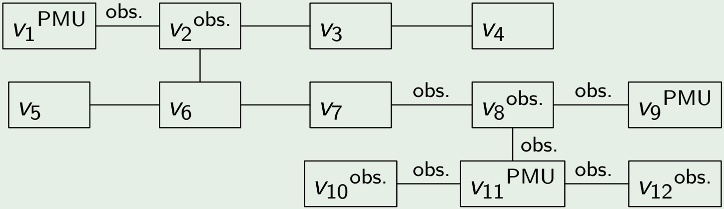

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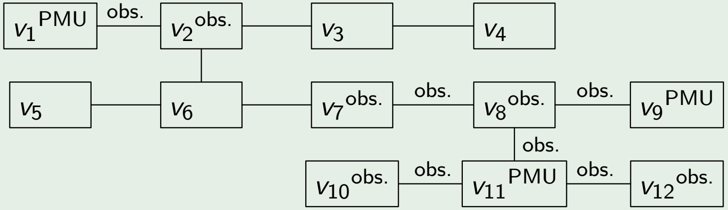

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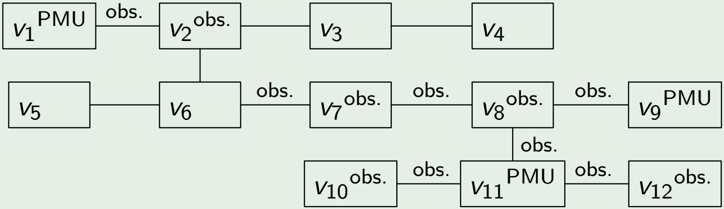

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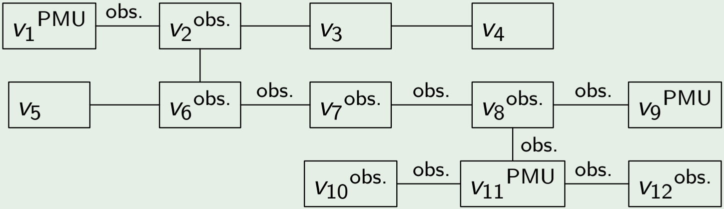

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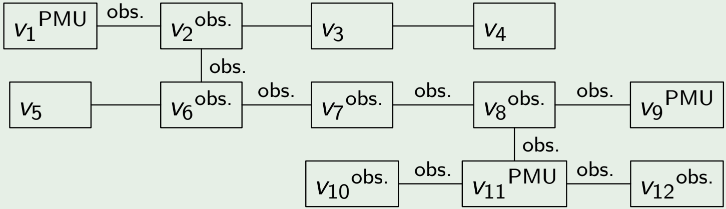

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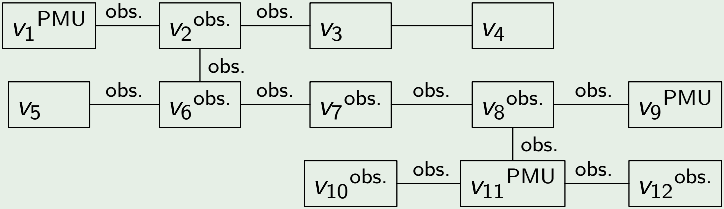

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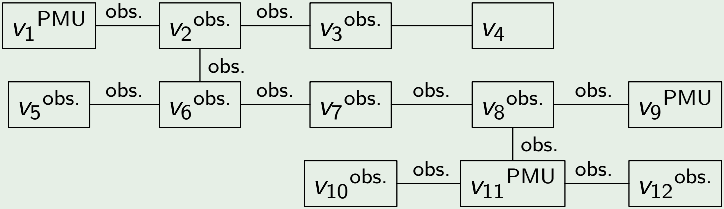

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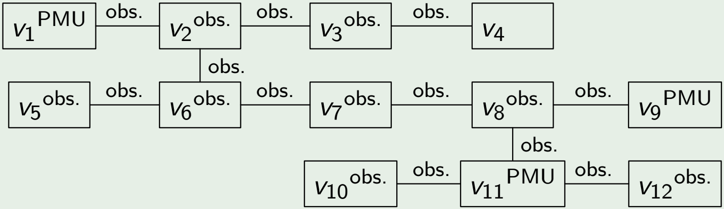

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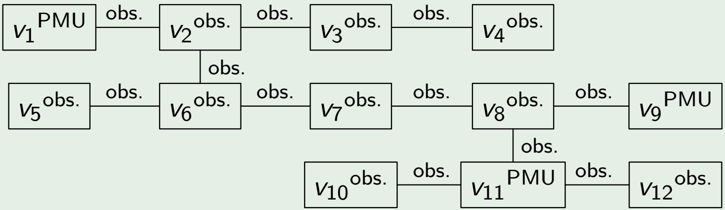

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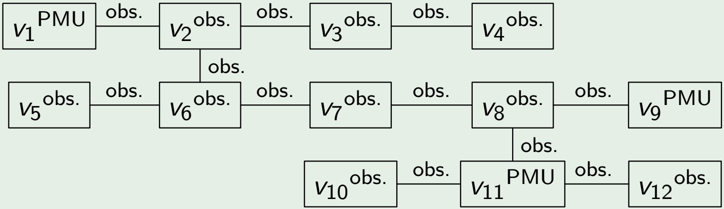

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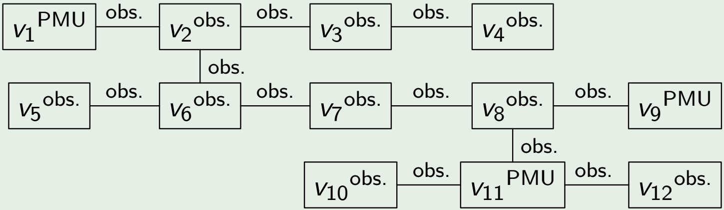

$$I_G^P = \langle X_1X_2X_3X_4, X_5X_6X_7X_8X_9, X_{10}X_{11}X_{12} \rangle$$

Alegraic Implications

Summary

For any ideal I, we have

$$I \text{ is a CI} \implies I \text{ is Gor.} \implies I \text{ is CM} \implies I \text{ is unmixed}$$

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Claim

In general, for abitrary graphs G,

I_G^P is a CI $\not\Leftarrow I_G^P$ is Gor. $\not\Leftarrow I_G^P$ is CM $\not\Leftarrow I_G^P$ is unmixed

Summery

Theorem (Villareal)

Let G be a tree. The following conditions are equivalent:

1. (i) *G is unmixed with respect to vertex covers.*
2. (ii) *G is a suspension of a subtree G'.*
3. (iii) *Every vertex of deg. ≥ 2 is adjacent to exactly 1 vertex of deg. 1.*
4. (iv) *I_G is Cohen-Macaualay.*

Theorem (CGHMS)

Let G be a tree. The following conditions are equivalent:

1. (i) *G is unmixed with respect to PMU covers.*
2. (ii) *G is obtained by "nicely linking" a sequence of paths.*
3. (iii) *Every vertex of deg. ≥ 3 is adjacent to exactly 2 vertices of deg. ≤ 2.*
4. (iv) *I_G^P is Cohen-Macaulay.*
5. (v) *I_G^P is Gorenstein and Complete Intersection.*

Counterexamples

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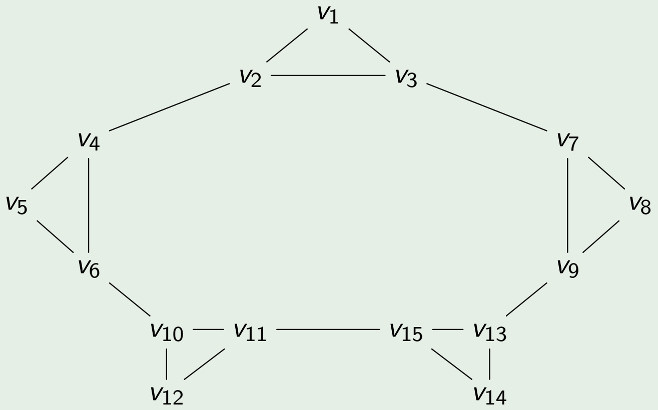
Example (I_G^P can be Gorenstein but not a Complete Intersection)



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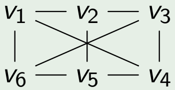

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Example (I_G^P can be Unmixed but not Cohen-Macaulay)

$$
\begin{array}{ccccccccc}
v_1 & — & v_2 & — & v_3 & — & v_4 & — & v_5 \\
 & & | & & & & | & & \\
v_6 & — & v_7 & — & v_8 & — & v_9 & — & v_{10}
\end{array}
$$

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