

Bernstein-Sato polynomials
on singular rings in
char. p.

§ DIFFERENTIAL OPERATORS:

k -field

R -finitely gen'd k -alg

• $\text{End}_k(R)$ is an $R \otimes_k R$ -module

$$(a \otimes b) \cdot \varphi = [x \mapsto a\varphi(bx)]$$

$$\bullet \mathcal{J} := (1 \otimes v - v \otimes 1 \mid v \in R)$$

Def: The ring of k -linear diff op's
on R

$$D_R := \left\{ \varphi \in \text{End}_k(R) \mid \exists n \gg 0 \right. \\ \left. \mathcal{J}^n \cdot \varphi = 0 \right\}$$

$D_R \subseteq \text{End}_k(R)$ is a subring

Key example: $R = \mathbb{C}[x_1, \dots, x_n]$

$$D_R = \left\{ \sum_{\alpha} f_{\alpha} \left(\frac{\partial}{\partial x_1} \right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n} \right)^{\alpha_n} \mid \alpha \in \mathbb{N}_0^n \right. \\ \left. f_{\alpha}(x) \in R \right\}$$

§ BERNSTEIN-SATO POLYNOMIAL:

$$R := \mathbb{C}[x_1, \dots, x_n]$$

Thm: (Bernstein, Sato)

let $f \in R$. Then $0 \neq b(s) \in \mathbb{C}[s]$,
 $\exists P(s) \in \mathbb{D}_R[s]$ s.t.

$$P(t) \cdot \underbrace{f^{t+1}}_{\substack{\uparrow \\ \text{in } R_f}} = b(t) \cdot f^t \quad \forall t \in \mathbb{Z} \quad (*)$$

Remk: c.f. Alvarez-Monster, Hernández.

[Jeffrey, Núñez-Betancourt, Teixeira, Witt]

i.e.:

$$0 \neq (b(s) \in \mathbb{C}[s] \mid \exists P(s) \in \mathbb{D}_R[s] \text{ s.t. } * \text{ holds})$$

Def: The BS-poly $b_f(s)$ of f is the monic generator of this ideal.

Examples:

(i) $f = x_1$

$$\frac{\partial}{\partial x_1} \cdot x_1^{t+1} = (t+1) \cdot x_1^t \quad \forall t \in \mathbb{Z}$$

$$\Rightarrow b_{x_1}(s) = (s+1).$$

$$b_f(s) \in \mathbb{Q}[s]$$

$$b_f(s) = (s-a_1) \dots (s-a_r)$$

$$(ii) \begin{bmatrix} x & y \\ u & v \end{bmatrix}$$

$$\left(\frac{\partial}{\partial x} \frac{\partial}{\partial v} - \frac{\partial}{\partial u} \frac{\partial}{\partial y} \right) \cdot (xv - uy)^{t+1} \\ = (t+1)(t+2) (xv - uy)^t$$

$$b_{xv-uy}(s) = (s+1)(s+2).$$

§ GENERALIZATION 1: R - \mathbb{C} -algebra
(need not be reg)

Def.: We say $f \in R$ is BS-good
if $\exists 0 \neq b(s) \in \mathbb{C}[s] \Rightarrow P(s) \in \mathcal{D}_R[s]$
s.t. $P(t) \cdot f^{t+1} = b(t) f^t \quad \forall t \in \mathbb{Z}$

Counterexamples: R -graded

• Bernstein-Gelfand-Gelfand $R = \frac{\mathbb{C}[x, y, z]}{(x^2 + y^2 + z^2)}$

• Mallory '20

Thm: (Álvarez-Montaner, Hucke, Niñez-Betancourt)

Suppose $R \oplus S := \mathbb{C}[x_1, \dots, x_n]$.

Then all $f \in R$ are BS-good.

Pf idea:

Know: $\exists P(s) \in D_S[s]$ s.t.

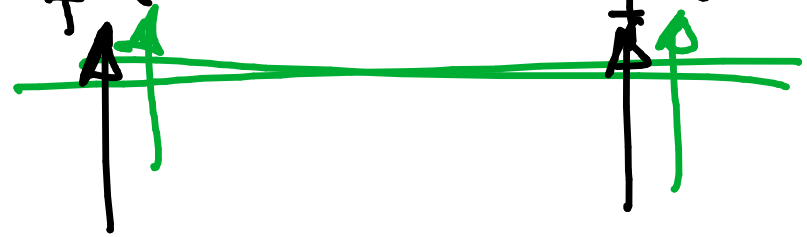
$$P(t) \cdot f^{t+1} = b_f^S(t) \cdot f^t \quad \forall t \in \mathbb{Z}.$$

$$\begin{array}{ccc}
 S & \xrightarrow{P} & S \\
 \uparrow i & & \downarrow \beta \\
 R & \xrightarrow{\tilde{P}_\beta} & R
 \end{array}$$

- $P_\beta \in D_R$
- $P_\beta(t) \cdot f^{t+1} = \underline{b_f^S(t)} \cdot f^t \quad \forall t \in \mathbb{Z}$

Cor: (of Pf)

$b_f^R(s)$ divides $b_f^S(s)$.



GENERALIZATION 2:

$$R = k[x_1, \dots, x_n],$$

k -perfect, char. $p > 0$.

• Mustată, Bittoun, QG (general σ)

• D_R :

$$\varphi \in D_R \Leftrightarrow \exists e \gg 0 \text{ s.t. } \mathcal{J}^{[pe]} \cdot \varphi = 0$$

$$\Leftrightarrow \exists e \gg 0 \forall r \in R, \varphi r^{pe} = r^{pe} \varphi$$

$$\Leftrightarrow \exists e \gg 0 \varphi \in \text{End}_{R^{pe}}(R).$$

$$(r^{pe} - r^{pe} \sigma) / (r \in R)$$

$$D_R = \bigcup_{e=0}^{\infty} D_R^e,$$

$$D_R^e = \text{End}_{R^{pe}}(R)$$

↑ diff op's of level e .

• Given $f \in R, e \geq 0$

$$B_f(p^e) := \{n \geq 0 \mid f^n \notin D_R^e \cdot f^{n+1}\} \subseteq \mathbb{N}_0$$

Properties:

$$(i) p^e \leq n \in B_f(p^e)$$

$$\Rightarrow n - p^e \in B_f(p^e)$$

(ii)

$$\dots \supseteq B_f(p^e) \supseteq B_f(p^{e+1}) \supseteq \dots$$

Fact (▲) $\exists k > 0$ s.t.

$$\left[\#(B_f(p^e) \cap [0, p^e)) \leq k \cdot \forall e \right]$$

$$D_R = \bigcup_{e=0}^{\infty} D_R^e, \quad D_R^e = \text{End}_{R^{pe}}(R)$$

↑ diff op's of level e .

Def: The BS-roots of f are

• Given $f \in R, e \geq 0$

$$BS(f) := \left\{ p\text{-lim}_{e \rightarrow \infty} (v_e) \mid v_e \in B_f(p^e) \right\} \subseteq \mathbb{Z}_p$$

↑
in fact $\subseteq \mathbb{Z}_{(p)}$

$$B_f(p^e) := \{ n \geq 0 \mid f^n \in D_R^e \cdot f^{n+1} \} \subseteq \mathbb{N}_0$$

▲ $\Rightarrow \#BS(f) < \infty$.

Example: $f = x^2 + y^3, p \equiv 1 \pmod{3}$

$$B_f(p^e) = \left\{ \frac{5}{6}(p^e - 1), p^e - 1 \right\} + p^e \mathbb{N}_0$$

$$\Rightarrow BS(f) = \left\{ -\frac{5}{6}, -1 \right\}$$

$$\left. \begin{array}{l} b_f(s) \\ \hline (s + \frac{5}{6})(s+1)(s + \frac{7}{6}) \end{array} \right\}$$

Properties:

(i) $p^e \leq n \in B_f(p^e)$

$$\Rightarrow n - p^e \in B_f(p^e)$$

(ii)

$$\dots \supseteq B_f(p^e) \supseteq B_f(p^{e+1}) \supseteq \dots$$

$$f = x^2 + y^2 + z^2$$

$$b_f(s) = (s+1) \left(s + \frac{3}{2}\right)$$

$$BS(f) = \{-1\}.$$

minimal exponent

§ RESULTS: (j.t. with
Jeffries, Núñez-Betancourt)

k -char. $p=0$, perfect.

R - k -algebra.

Def: $f \in R$ is BS-good if $\exists k > 0$
(s.t. $\#(B_f(p^e) \cap [0, p^e]) \leq k \forall e \geq 0$)

Thm: (Jeffries, Núñez-Betancourt, QG)
If $R = \bigoplus_{n=0}^{\infty} R_n$ graded over k
with finite F-representation type.
Then all $f \in R$ are BS-good.

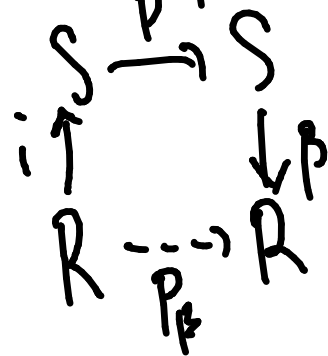
Thm: (J, NB, QG) R is F-split, $f \in R$
a BS-good element. Then
 $BS(f) \subseteq \mathbb{Z}(p)$.

Thm: (\exists , NB, QG)

$(R \otimes S = k[x_1, \dots, x_d])$. Then all $f \in R$ are BS-good.

Pf: $B_f^R(p^e) \subseteq B_f^S(p^e)$

$n \notin B_f^S(p^e) \iff \exists P \in D_S^e$ s.t. $P \cdot f^{n+1} = f$



- $P_\beta \in D_R^e$
 - $P_\beta \cdot f^{n+1} = f$
- $\implies n \notin B_f^R(p^e)$

Cor: (\exists proof)

$B_f^R \subseteq B_f^S$

D^e — ann $\mathcal{J}^{[pe]}$

D^{n+1} — ann \mathcal{J}^{n+1}

$R = k[x_1, \dots, x_d]$

$(\text{--- } x_1 \text{---}, \dots, \dots \text{--- } x_d \text{---})$

$f \in \mathbb{R}$

Thm: (Bitom)

$$\left[\text{BS}(f) = \sim F \int (f) \cap (0,1) \cap \mathbb{Z}(p) \right]$$