Bernstein-Sato polynomials on singular rings in char. p.

§ DIFFERENTIAL OPERATORS:

k - field

R – ftely gen'd k-alg

- $\mathrm{End}_k(R)$ is an $R \otimes_k R$-module

$(a \otimes b) \cdot \varphi = [x \mapsto a\varphi(bx)]$

- $J := (1 \otimes r - r \otimes 1 \mid r \in R)$

Def: The ring of k-linear diff op's on R

$D_R := \{ \varphi \in \mathrm{End}_k(R) \mid \exists n \gg 0 \; J^n \cdot \varphi = 0 \}$

$D_R \subseteq \mathrm{End}_k(R)$ is a subring

Key example: $R = \mathbb{C}[x_1, \dots, x_n]$

$D_R = \left\{ \sum_\alpha' f_\alpha \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} \;\middle|\; \begin{array}{l} \alpha \in \mathbb{N}_0^n \\ f_\alpha(x) \in R \end{array} \right\}$

§ BERNSTEIN-SATO POLYNOMIAL:

$R := \mathbb{C}[x_1, \ldots, x_n]$

Thm: (Bernstein, Sato)

Let $f \in R$. Then $0 \neq b(s) \in \mathbb{C}[s]$, $\exists P(s) \in D_R[s]$ s.t.

$$P(t) \cdot f^{t+1} = b(t) \cdot f^t \quad \forall t \in \mathbb{Z} \quad (*)$$

in R_f

Rmk: c.f. Álvarez-Montaner, Hernández, Jeffries, Núñez-Betancourt, Teixeira, Witt

i.e.

$$0 \neq \left(b(s) \in \mathbb{C}[s] \;\middle|\; \begin{array}{l} \exists P(s) \in D_R[s] \\ \text{s.t. } * \text{ holds} \end{array} \right) \underset{\text{ideal}}{\subseteq} \mathbb{C}[s].$$

Def: The BS-poly $b_f(s)$ of f the monic generator of this ideal.

$b_f(s) \in \mathbb{Q}[s]$

$b_f(s) = (s - a_1)^{e_1} \cdots (s - a_r)^{e_r}$

Examples:

(i) $f = x_1$

$$\frac{\partial}{\partial x_1} \cdot x_1^{t+1} = (t+1) \cdot x^t \quad \forall t \in \mathbb{Z}$$

$$\Rightarrow b_{x_1}(s) = (s+1).$$

(ii) $\begin{bmatrix} x & y \\ u & v \end{bmatrix}$

$$\left(\frac{\partial}{\partial x}\frac{\partial}{\partial v} - \frac{\partial}{\partial u}\frac{\partial}{\partial y}\right)\cdot (xv-uy)^{t+1}$$

$$= (t+1)(t+2)\,(xv-uy)^t$$

$$b_{xv-yu}(s) = (s+1)(s+2).$$

§ GENERALIZATION 1 : R - $\mathbb{C}$-algebra (need not be reg)

Def: We say $f \in R$ is BS-good if $\exists\, 0 \neq b(s) \in \mathbb{C}[s]$ $\exists\, P(s) \in D_R[s]$ s.t. $P(t)\cdot f^{t+1} = b(t) f^t \;\; \forall t \in \mathbb{Z}$.

Counterexamples: R - graded

- Berstein - Gelfand - Gelfand $R = \dfrac{\mathbb{C}[x,y,z]}{(x^3+y^3+z^3)}$
- Mallory '20

Thm: (Álvarez-Montaner, Huneke, Núñez-Betancourt)

Suppose $R \underset{\oplus}{\subseteq} S := \mathbb{C}[x_1, \ldots, x_n]$.

Then all $f \in R$ are BS-good.

Pf idea:

Know: $\exists P(s) \in D_S[s]$ s.t.

$$P(t) \cdot f^{t+1} = b_f^S(t) \cdot f^t \quad \forall t \in \mathbb{Z}.$$

$$\begin{tikzcd} S \arrow[r, "P"] & S \arrow[d, "\beta"] \\ R \arrow[u, "i"] \arrow[r, dashed, "P_\beta"'] & R \end{tikzcd}$$

- $P_\beta \in D_R$
- $P_\beta(t) \cdot f^{t+1} = b_f^S(t) \cdot f^t \quad \forall t \in \mathbb{Z}$

Cor: (of Pf)

$b_f^R(s)$ divides $b_f^S(s)$.

GENERALIZATION 2 :

$R = k[x_1, \dots, x_n]$,

k - perfect, char. $p > 0$.

- Mustaţă, Bitoun, QG (general α)

- D_R :

$(1 \otimes r^{p^e} - r^{p^e} \otimes 1 \mid r \in R)$

$\varphi \in D_R \iff \exists e \gg 0$ s.t. $\mathcal{J}^{[p^e]} \cdot \varphi = 0$

$\iff \exists e \gg 0 \ \forall r \in R, \ \varphi r^{p^e} = r^{p^e} \varphi$

$\iff \exists e \gg 0 \ \varphi \in \mathrm{End}_{R^{p^e}}(R)$.

$D_R = \bigcup_{e=0}^{\infty} D_R^e$, $\quad D_R^e = \mathrm{End}_{R^{p^e}}(R)$

diff op's of level e.

- Given $f \in R$, $e \geq 0$

$B_f(p^e) := \{ n \geq 0 \mid f^n \notin D_R^e \cdot f^{n+1} \}$

$\subseteq \mathbb{N}_0$

Properties:

(i) $p^e \leq n \in B_f(p^e)$

$\Rightarrow n - p^e \in B_f(p^e)$

(ii)

$\dots \supseteq B_f(p^e) \supseteq B_f(p^{e+1}) \supseteq \dots$

Fact (▲) $\exists K>0$ s.t.

$$\left[\#\left(B_f(p^e)\cap[0,p^e)\right)\leq K\ \forall e\right]$$

Def: The BS-roots of f are

$$BS(f) := \left\{ p\text{-}\lim_{e\to\infty}(\nu_e) \;\middle|\; \nu_e \in B_f(p^e) \right\} \subseteq \mathbb{Z}_p$$

in fact $\subseteq \mathbb{Z}_{(p)}$

▲ $\Rightarrow \# BS(f) < \infty$.

Example: $f = x^2+y^3$, $p \equiv 1 \mod 3$

$b_f(s) = (s+\frac{5}{6})(s+1)(s+\frac{7}{6})$

$$B_f(p^e) = \left\{\frac{5}{6}(p^e-1), p^e-1\right\} + p^e\mathbb{N}_0$$

$$\Rightarrow BS(f) = \left\{-\frac{5}{6}, -1\right\}$$

$$D_R = \bigcup_{e=0}^{\infty} D_R^e, \quad D_R^e = \mathrm{End}_{R^{p^e}}(R)$$

↑ diff op's of level e.

- Given $f\in R$, $e\geq 0$

$$B_f(p^e) := \{ n\geq 0 \mid f^n \notin D_R^e \cdot f^{n+1} \} \subseteq \mathbb{N}_0$$

Properties:

(i) $p^e \leq n \in B_f(p^e) \Rightarrow n - p^e \in B_f(p^e)$

(ii)

$$\cdots \supseteq B_f(p^e) \supseteq B_f(p^{e+1}) \supseteq \cdots$$

$f = x^2+y^2+z^2$

$b_f(s) = (s+1)(s+\frac{3}{2})$ [minimal exponent]

$BS(f) = \{-1\}$.

§ RESULTS: (jt. with Jeffries, Núñez-Betancourt)

k - char. $p>0$, perfect.

R - k-algebra.

Def: $f \in R$ is BS good if $\exists K>0$ s.t. $\#\left(B_f(p^e) \cap [0,p^e)\right) \le K \;\; \forall e \ge 0$

Thm: (Jeffries, Núñez-Betancourt, QG) If $R = \bigoplus_{n=0}^{\infty} R_n$ graded over $R_0 = k$ with finite F-representation type. Then all $f \in R$ are BS-good.

Thm: (J, NB, QG) R is F-split, $f \in R$ a BS-good element. Then
$$BS(f) \subseteq \mathbb{Z}_{(p)}.$$

Thm: (J, NB, QG)

[$R \underset{\oplus}{\subseteq} S = k[x_1, \dots, x_d]$. Then all $f \in R$ are BS-good.]

Pf: $B_f^R(p^e) \subseteq B_f^S(p^e)$

$\mathfrak{n} \notin B_f^S(p^e) \Leftrightarrow \exists P \in D_S^e$ s.t. $P \cdot f^{n+1} = f$

$$\begin{tikzcd} S \arrow[r, "P"] & S \arrow[d, "\beta"] \\ R \arrow[u, "i"] \arrow[r, dashed, "P_\beta"'] & R \end{tikzcd}$$

- $P_\beta \in D_R^e$
- $P_\beta \cdot f^{n+1} = f$

$\Rightarrow \mathfrak{n} \notin B_f^R(p^e)$

Cor: (of proof)

[$BS^R(f) \subseteq BS^S(f)$.]

D^e — ann $J^{[p^e]}$

$D^{\leq n}$ — ann J^{n+1}

$R = k[x_1, \dots, x_d]$ $\quad$ $(1 \otimes x_1 - x_1 \otimes 1, \dots, 1 \otimes x_d - x_d \otimes 1)$

$f \in \mathbb{R}$

Thm:- (Bitom)

$$\left[BS(f) = -FJ(f) \cap [0,1] \cap \mathbb{Z}(p) \right]$$