Extremal Singularities in Positive Characteristic

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(and with Anna Brosowsky, Tim Ryan, and Karen Smith)

<u>Motivating Question</u>: What is the most singular possible reduced hypersurface (of degree d) in char $p>0$?

<u>One answer</u>: Those with minimal F-pure threshold.

<u>Def</u>: Let $f \in k[x_1, \dots, x_n]$, k char $p>0$, $f \in \mathfrak{m}$. Then

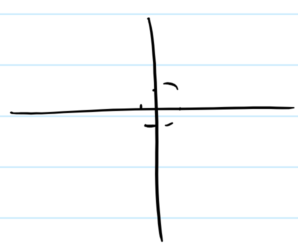
$$\mathrm{fpt}_{\mathfrak{m}}(f) := \sup\left\{\frac{N}{p^e} \;\middle|\; f^N \notin \mathfrak{m}^{[p^e]} = \langle x_1^{p^e}, \dots, x_n^{p^e}\rangle\right\}$$
$$= \inf\left\{\frac{N}{p^e} \;\middle|\; f^N \in \mathfrak{m}^{[p^e]}\right\}$$

<u>Remarks</u>: Say f is degree d (n vars) k char $p>0$. Then

- $0 < \mathrm{fpt}(f) \le 1$
- In fact, you can show: $1/d \le \mathrm{fpt}(f) \le n/d$
- In general, the smaller the $\mathrm{fpt}(f)$, the more singular f is
- If f is smooth, the $\mathrm{fpt}(f) = 1$
- for any r, $\mathrm{fpt}(f^r) = \frac{\mathrm{fpt}(f)}{r}$

<u>Ex</u> $\mathrm{fpt}(x^d) = \frac{1}{d}$

<u>Ex</u>



$xy=0$

$\mathrm{fpt}(xy)=1$

$y^2-x^3=0$

$$\mathrm{fpt}(y^2-x^3) = \begin{cases} 1/2 & p=2 \\ 2/3 & p=3 \\ 5/6 & p \equiv 1 \bmod 6 \\ 5/6 - 1/6p & p \equiv 5 \bmod 6 \end{cases}$$

Thm (KK-SSVW) If $f \in k[x_1, \dots, x_n]$, k char $p > 0$, f homogeneous & reduced of degree d, then

$$\mathrm{fpt}(f) \geq \frac{1}{d-1}$$

In addition $\mathrm{fpt}(f) = \frac{1}{d-1} \iff d = p^e + 1$ for some e and $f \in \mathfrak{m}^{[p^e]} = (x_1^{p^e}, \dots, x_n^{p^e})$

$\iff$

$(*)$ $f = x_1^{p^e} L_1 + \dots + x_n^{p^e} L_n$ such that each L_i is linear in $x_1, \dots, x_n$

Def We'll say f is a Frobenius form if f has form $*$

(We'll say f or $\mathbb{V}(f)$ is an extremal singularity / extremal hypersurface if $\mathrm{fpt}(f) = \frac{1}{d-1}$

Ex (of $*$) $f = x^3 + y^3 + z^3 + w^3$ in char 2.

(or any cubic surface which is not F-pure is also of form $*$)

Key observation: If f is a Frobenius form, we can write:

$$f = \begin{bmatrix} x_1^{p^e} & \cdots & x_n^{p^e} \end{bmatrix} \underset{\text{n×n matrix w/ coeff in k}}{A} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \leftarrow$$

A is unique

Also, if I consider a change of coordinates φ

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \longmapsto \overset{n \times n}{g} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

Then

$$\varphi(f) = [x_1^{p^e} \cdots x_n^{p^e}] (g^{tr})^{[p^e]} A\, g \begin{bmatrix} x_1 \\ \vdots \end{bmatrix}$$

all coords raised to p^e

Then $$\varphi(f) = [x_1^{p^e} \cdots x_n^{p^e}](g^{tr})^{[p^e]} A\, g \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

__Def__ A Frobenius form is __rank r__ if the associated matrix A has rank r.

__Thm 2__ [KK-SSVW] (Beauville '90 $n \geq 4$) If f is a Frobenius form in n variables of rank n, then

$$f \sim x_1^{p^e+1} + \cdots + x_n^{p^e+1} \quad \left(\text{ie } A = \begin{pmatrix} 1 & 0 & \cdots \\ & \ddots & \\ 0 & & 1 \end{pmatrix}\right)$$

__Thm 3__ (KK-SSVW) There are finitely many Frobenius forms in any bounded degree & # of variables. (up to change of coordinates)

More specifically, there is a bijection

$$\left\{ \begin{array}{c} \text{the partitions} \\ \text{of } n \end{array} \right\} \longmapsto \left\{ \begin{array}{c} \text{Frobenius forms} \\ \text{w/c are nondegenerate} \\ \text{in } n \text{ variables and} \\ \text{degree } d = p^e+1 \end{array} \right\}$$

__Sketch__: First, by induction (& a lot of linear algebra)

uses Thm 2

show that without loss of generality, the matrix A associated to a Frob. form f has

call A "sparse" (• at most one nonzero entry in every row/col. (& it is a 1))

However, we know that (for example) the following two matrices give linearly equiv. Frob. forms:

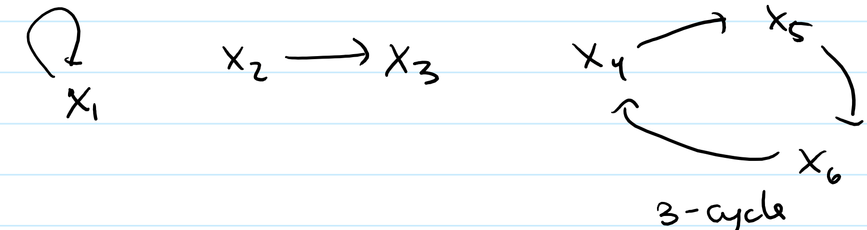
① $$\begin{bmatrix} 0&0&0&0&1 \\ 0&0&0&1&0 \\ 0&1&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{bmatrix} \sim ② \begin{bmatrix} 0&0&0&0&1 \\ 0&0&0&1&0 \\ 1&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{bmatrix}$$

$$x_1^{p^e}x_5 + x_2^{p^e}x_4 + x_3^{p^e}x_2 \qquad\qquad x_1^{p^e}x_5 + x_2^{p^e}x_4 + x_3^{p^e}x_1$$

Def Given a Frobenius form f in "sparse" form, associate to f a directed graph Γ_f on n vertices by $x_i \longrightarrow x_j \iff x_i^{p^e} x_j$ is a term of f

Ex ① $x_1 \to x_5$, $x_3 \to x_2 \to x_4$ ② $x_3 \to x_1 \to x_5$, $x_2 \to x_4$

③ $x_1^{p^e+1} + x_2^{p^e} x_3 + x_4^{p^e} x_5 + x_5^{p^e} x_6 + x_6^{p^e} x_4$



Note: Γ_f can have
- loops
- directed chains
- directed cycles

(by assumptions on A)

Side note: A is the directed adjacency matrix of Γ_f.

Note: If Γ_f has an ℓ-cycle, then

$$f = \underbrace{x_1^{p^e} x_2 + x_2^{p^e} x_3 + \cdots + x_{\ell-1}^{p^e} x_\ell + x_\ell^{p^e} x_1}_{\text{this is a full rank FF in } \ell \text{ variables!}} + f'(x_{\ell+1}, \dots, x_n)$$

So by Thm 2, there is a change of coordinates on $x_1, \dots, x_\ell$ so that

$$f \sim x_1^{p^e+1} + \cdots + x_\ell^{p^e+1} + f'(x_{\ell+1}, \dots, x_n)$$

So, WLOG, f is equivalent to some g so that Γ_g has only
- loops
- chains

Now associate a partition by the # of vertices in each component of Γ_g

Ex ①② $2+3=5$ $\cdot \to \cdot \quad \cdot \to \cdot \to \cdot$

③ $\Omega\,\Omega\,\Omega\,\Omega\ \cdot \to \cdot$ $1+1+1+1+2=6$

Final step: Clearly if f_1 & f_2 have the same partition then $f_1 \sim f_2$

uses some algebra [We also show that if $f_1 \sim f_2$, then f_1 & f_2 have the same partition

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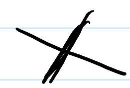
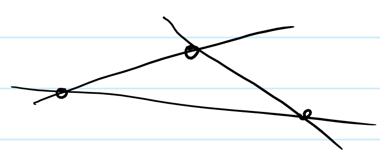
Geometry Suppose $X = \mathbb{V}(f)$ where f is a Frobenius form. & suppose ℓ_1 & ℓ_2 are intersecting lines on X. Let Λ be the plane they span. If $\Lambda \subseteq X$ then $\Lambda \cap X$ is either

- a p^e-fold line intersecting another line 
- "perfect" star  p^e+1

Consequence: X has no "triangles"



Ex We know smooth cubic surfaces have 27 lines. But in char 2, the smooth cubic surface $x^3+y^3+z^3+w^3$ has no triangles $\rightarrow$ every intersection point is an Eckardt point

This is the only smooth cubic surface in any char with no triangles!

(ongoing work (B-RS)) studying extremal surfaces (4 variables)