Tate-like complexes / Applications to DG structures.

Def: Let $(F_\bullet, d_\bullet)$ be a complex. $F_\bullet$ has the structure of an associative DG algebra if there's an associative product,

$$\cdot : F_i \otimes F_j \longrightarrow F_{i+j}$$

s.t. $d_{i+j}(f_i \cdot f_j) = d_i(f_i)\cdot f_j + (-1)^i f_i \cdot d_j(f_j)$ $(f_i \in F_i)$.

Also: $f_i \cdot f_j = (-1)^{ij} f_j \cdot f_i$

Moreover, $F_\bullet$ exhibits Poincaré duality if $F_m \cong R$ $(m = \text{length } F)$, and

$$F_i \otimes F_{m-i} \longrightarrow F_m \text{ is perfect}$$

$\left(\text{i.e., } F_i \xrightarrow{\sim} F_{m-i}^*, \quad F_{m-i} \xrightarrow{\sim} F_i^*\right)$

Canonical Example: Koszul complex

$F = Rf_1 \oplus \cdots \oplus Rf_n$,

$\psi : F \longrightarrow R$.

$K_i = \Lambda^i F$

$d : \Lambda^i F \longrightarrow \Lambda^{i-1} F$

$f_{j_1} \wedge \cdots \wedge f_{j_i} \longmapsto \sum (-1)^{\ell+1} \psi(f_{j_\ell}) f_{j_1} \wedge \cdots \wedge \widehat{f_{j_\ell}} \wedge \cdots \wedge f_{j_i}$ (excluded)

Mult: $\Lambda^i F \otimes \Lambda^j F \longrightarrow \Lambda^{i+j} F$

Can check: satisfies product rule.
Also exhibits Poincaré Duality.

Theorem (Buchsbaum–Eisenbud, 1977)
Let R/I be a ring $\operatorname{pd}_R R/I = 3$. Then the minimal free res. of R/I admits the structure of an assoc. DGA.

Pf: Let $(F_\bullet, d_\bullet) \to R/I$ be the MFR. Consider the following cx:

$$D_2(F) : F_1 \otimes F_2 \xrightarrow{\partial_3} F_2 \oplus \Lambda^2 F_1 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} R$$

$\partial_3(f_1 \otimes f_2) = d_1(f_1)\cdot f_2 - f_1 \wedge d_2(f_2)$

$\partial_2|_{F_2} = d_2$

$\partial_2(f_1 \wedge f_1') = d(f_1) f_1' - d_1(f_1') f_1$

$\partial_1 = d_1$

Choose any comparison map:

$c_\bullet : D_2(F) \longrightarrow F$

w/ $\psi_1 : \Lambda^2 F_1 \longrightarrow F_2$, $\psi_2 : F_1 \otimes F_2 \longrightarrow F_3$ } use these to define the product.

d_3 is injective. Apply d_3 to the difference

$\psi_2(\psi_1(f_1 \wedge f_1') \otimes f_1'') - \psi_2(f_1 \otimes \psi_1(f_1' \wedge f_1''))$

$a \cdot b := \psi_1(a \wedge b)$

$D_2(F)$ is often called the symmetric square complex.

Tate-like Complexes.

Let $(F_\bullet, d_\bullet)$ be a complex, with $F_0 = R$.

$$N_a(F_i) := \begin{cases} \Lambda^a F_i, & i \text{ odd}, \\ D_a(F_i), & i \text{ even}. \end{cases}$$

Notice:

$\partial : N_a(F_i) \longrightarrow F_{i-1} \otimes N_{a-1}(F_i)$, factoring as $N_a(F_i) \xrightarrow{\Delta} F_i \otimes N_{a-1}(F_i) \xrightarrow{d_i \otimes 1} F_{i-1} \otimes N_{a-1}(F_i)$

Ex: f_1, f_2, f_3. image under $(1,2)$-comult.

$f_1 \wedge f_2 \wedge f_3 \longmapsto f_1 \otimes f_2 \wedge f_3 - f_2 \otimes f_1 \wedge f_3 + f_3 \otimes f_1 \wedge f_2$

Def: A degree k Tate-like complex is complex $D_k(F_\bullet)$ with:

$$D_k(F_\bullet)_j := \bigoplus_{\substack{(a_1, \ldots, a_n) \\ \sum a_i = k \\ \sum_i i a_i = j}} N_{a_1}(F_1) \otimes \cdots \otimes N_{a_n}(F_n)$$

$$\partial : D_k(F)_j \longrightarrow D_k(F)_{j-1}$$

induced by ∂ as above.

$D_2(F)$ is exactly the symmetric square

- Buchsbaum–Eisenbud,
- Kustin–Miller (for grade 4 Gor. ideals)
- Matrix factorization (Kustin).
- Weyman–Lebelt complexes

Setup: $\mathfrak{a} \subseteq I$ ($\mathfrak{a}$: grade 3 CI; I: grade 4 Gorenstein ideal.)

$M_\bullet \to R/I$ is associative DGA MFR exhibiting Poincaré duality.

$K_\bullet \to R/\mathfrak{a}$ length 3 Koszul cx.

$\alpha : K_\bullet \longrightarrow M_\bullet$ comparison map ext. the identity.

Thm (Kustin–Miller, 83)
Let I' be tightly double linked to I. Then, there exists an $r \in R$ and a complex $F(\alpha, r)$ ("Big From Small complex") such that $F(\alpha, r)$ is a res. of R/I'.

Q: Does $F(\alpha, r)$ admit the struct of an assoc. DGA?

A: Yes, if $1/2 \in R$. (Kustin, 94)

based on work by Palmer.

This is a main step in the proof that the MFR of a grade 4 ACI is an assoc. DGA.

The method of proof uses "complete higher order mult" on M.

I construct something slightly weaker, but good enough.

Idea: Build maps

$X : \Lambda^2 M_1 \longrightarrow M_2$,

$X^t : M_1 \otimes M_2 \longrightarrow M_3$.

induced by homotopies arising from morph. of Tate-like complexes. More precisely,

$c_\bullet : D_3(M) \longrightarrow K[-2]$ (acyclic)

$\Rightarrow c$ is nullhomotopic by some h.

$$D_3(M)_4 \xrightarrow{h_4} K_3 \xrightarrow{\sim} M_4, \quad \Lambda^2 M_1 \otimes M_2 \hookrightarrow D_3(M)_4, \quad \Phi : \Lambda^2 M_1 \otimes M_2 \to M_4$$

Look at $\Phi : \Lambda^2 M_1 \otimes M_2 \longrightarrow M_4$

Define X, X^t implicitly via:

$$X(\theta_1 \wedge \theta_1')\cdot \theta_2 := \Phi(\theta_1 \wedge \theta_1' \otimes \theta_2) =: \theta_1' \cdot X^t(\theta_1 \otimes \theta_2)$$

where $\theta_1, \theta_1' \in M_1$, $\theta_2 \in M_2$.

Thm (–, 2020): X and X^t satisfy a whole bunch of convenient identities.

These maps can be used to endow $F(\alpha, r)$ with the structure of an assoc. DGA exh. Poincaré Duality.