Decompositions of Modules over Principal Subalgebras of Truncated Polynomial Rings

Kevin Harris

January 24, 2022

Outline

- Background
- Decomposition Theorem for PIDs
- Representation Matrices
- Clebsch-Gordan Problem
- Results
- Future Goals

Background

- Let $A = A_p^n$ denote the following truncated polynomial ring:

$$A_p^n = \frac{k[x_1, \ldots, x_n]}{(x_1^p, \ldots, x_n^p)}$$

where k is a field and $char(k) = p$ for some prime p.

- Let $R = k[u_\lambda]$ denote the subalgebra of A where $\lambda = (\lambda_1, \ldots, \lambda_n) \in k^n$, and $u_\lambda = \lambda_1 x_1 + \cdots + \lambda_n x_n$.

- Note that in R :

$$u_\lambda^p = (\lambda_1 x_1 + \cdots + \lambda_n x_n)^p = \lambda_1^p x_1^p + \cdots + \lambda_n^p x_n^p \equiv 0.$$

Decomposition of an A-module M over R

Fundamental Theorem for Modules over a PID

Let M be a finitely generated module over a principal ideal domain R. Then M is a direct sum of cyclic submodules. More precisely, there exist nonnegative integers h, m, irreducible elements $p_1, \ldots, p_m \in R$ and positive integers $t_1, \ldots, t_m$ such that

$$M \cong R/Rp_1^{t_1} \oplus \cdots \oplus R/Rp_m^{t_m} \oplus R^h.$$

Applying theorem to $k[x]/(x^p)$

- There is a natural epimorphism $f : k[x] \to k[x]/(x^p)$.
- Thus, any finitely generated $k[x]/(x^p)$-module M can also be considered as a $k[x]$-module with

$$M \cong k[x]/(p_1^{t_1}) \oplus \cdots \oplus k[x]/(p_m^{t_m}) \oplus k[x]^h.$$

- When viewing M as a $k[x]/(x^p)$-module, we have that $x^p M = 0$. This forces $h = 0$ and $x^p k[x]/(p_i^{t_i}) = 0$ for each i. So we have that

$$x^p \in (p_i^{t_i}) \Rightarrow (x^p) \subseteq (p_i^{t_i}) \Rightarrow (x) \subseteq (p_i) \Rightarrow (x) = (p_i) \ \ \forall\, i.$$

Decomposing M over $k[x]/(x^p)$

Decomposition Theorem for $k[x]/(x^p)$

Let M be a finitely generated module over $k[x]/(x^p)$ and $D_i = k[x]/(x^i)$ for any $1 \leq i \leq p$. Then there exists nonnegative integers m_i, $i = 1, \ldots, p$ such that

$$M \cong D_1^{m_1} \oplus D_2^{m_2} \oplus \cdots \oplus D_p^{m_p}.$$

- Is there a free part?
- Consider the map $\phi : k[x] \to k[u_\lambda]$ defined by $x \mapsto u_\lambda$.
- Note that ϕ is surjective and $\ker(\phi) = (x^p)$.
- Thus, by F.I.T., we have $D_p = k[x]/(x^p) \cong k[u_\lambda]$.
- Moreover, $D_i \cong k[u_\lambda]/(u_\lambda^i)$ for each $i = 1, \ldots, p$.

Question 1

Let M be a finitely generated A-module. If k is infinite, there are infinitely many choices for $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$. How does the decomposition of M change as λ changes? Does the decomposition always change?

Example 1

- Let $A = \frac{k[x_1,x_2]}{(x_1^2,x_2^2)}$. Consider the ideal $I = (x_1)$ of A. Clearly, I is an A-module. Note that I has the following k-basis: $\{x_1, x_1x_2\}$.

- Recall that $R = k[u_\lambda]$ is the subalgebra of A such that $u_\lambda = \lambda_1 x_1 + \lambda_2 x_2$ where $\lambda_1, \lambda_2 \in k$.

- Suppose $\lambda_1 = 1$ and $\lambda_2 = 0$. Then $R = k[x_1]$ so I needs two generators as an R-module, take $\{x_1, x_1x_2\}$. So we have that

$$\begin{aligned} I &= Rx_1 \oplus Rx_1x_2 \\ &\cong R/ann_R(x_1) \oplus R/ann_R(x_1x_2) \\ &\cong R/(x_1) \oplus R/(x_1) \\ &\cong k^2 \end{aligned}$$

Same Example

- Suppose $\lambda_1 = 0$ and $\lambda_2 = 1$. Then $R = k[x_2]$ so I only needs one generator as an R-module, take $\{x_1\}$. So we have that

$$\begin{aligned} I &= Rx_1 \\ &\cong R/ann_R(x_1) \\ &\cong R/\langle 0_R \rangle \\ &\cong R. \end{aligned}$$

- Notice we not only have a different decomposition of I as an R-module, but we can also see that I can be viewed as a free R-module with the right conditions for λ.

Example 2

- Use the same A from the previous example and let $I = (x_1, x_2)$. Note that I has the following k-basis: $\{x_1, x_2, x_1x_2\}$.

- Suppose $\lambda_1 = 1$ and $\lambda_2 = 0$. Then $R = k[x_1]$ so I needs two generators as an R-modules, take $\{x_1, x_2\}$. So we have that

$$\begin{aligned} I &= Rx_1 \oplus Rx_2 \\ &\cong R/ann_R(x_1) \oplus R/ann_R(x_2) \\ &\cong R/(x_1) \oplus R/\langle 0_R \rangle \\ &\cong k \oplus R. \end{aligned}$$

Same Example

- Suppose $\lambda_1 = 0$ and $\lambda_2 = 1$. Then $R = k[x_2]$ so I still needs two generators as an R-module, take $\{x_1, x_2\}$. So we have that

$$\begin{aligned} I &= Rx_1 \oplus Rx_2 \\ &\cong R/ann_R(x_1) \oplus R/ann_R(x_2) \\ &\cong R/\langle 0_R \rangle \oplus R/(x_2) \\ &\cong R \oplus k. \end{aligned}$$

- Notice how the decomposition didn't change even though the conditions on λ did.

Fact

The decomposition for this module will not change as λ changes. This is called a Module of Constant Jordan Type.

Let's look at it a different way

- Use the same A from the previous examples $(n = 2, p = 2)$. Consider the ideal $I = (x_1)$ of A.

- We know that $\{x_1, x_1x_2\}$ is a k-basis for I. What happens when we multiply both of those elements by $u_\lambda = \lambda_1 x_1 + \lambda_2 x_2$? We have

$$\begin{aligned} u_\lambda x_1 &\equiv \lambda_2 x_1 x_2 \\ u_\lambda x_1 x_2 &\equiv 0. \end{aligned}$$

- We can also view the module as a matrix by looking at the "representation matrix" of u_λ over I with respect to the basis shown above. We will denote it as $\left[u_\lambda\right]_I$.

$$\left[u_\lambda\right]_I = \begin{array}{c} \begin{array}{cc} x_1 & x_1x_2 \end{array} \\ \begin{bmatrix} 0 & 0 \\ \lambda_2 & 0 \end{bmatrix} \end{array}$$

What is a Representation Matrix?

- Let M be a finitely generated A-module with the following k-basis: $\{e_1, e_2, \ldots, e_m\}$.

- Suppose there exists $a_{ij} \in k$ such that

$$\begin{aligned} u_\lambda e_1 &= a_{11} e_1 + a_{21} e_2 + \cdots + a_{m1} e_m \\ u_\lambda e_2 &= a_{12} e_1 + a_{22} e_2 + \cdots + a_{m2} e_m \\ &\vdots \\ u_\lambda e_m &= a_{1m} e_1 + a_{2m} e_2 + \cdots + a_{mm} e_m. \end{aligned}$$

- Then the following matrix is the representation matrix of u_λ on M with respect to the k-basis $\{e_1, e_2, \ldots, e_m\}$:

$$[u_\lambda]_M = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{bmatrix}$$

Why do we need representation matrices?

- Since we know

$$u_\lambda = \lambda_1 x_1 + \cdots + \lambda_n x_n,$$

it immediately follows that

$$\left[u_\lambda\right]_M = \lambda_1 \left[x_1\right]_M + \cdots + \lambda_n \left[x_n\right]_M .$$

- Another way to look at the decomposition of an A-module M over R is through each of the representation matrices of x_i on M, denoted as $[x_i]_M$.
- One of our goals is to figure out what $[x_i]_{M \otimes N}$ looks like if we know $[x_i]_M$ and $[x_i]_N$.

Translating to One Variable

- Let M be a finitely generated A-module.

- Recall that there exists nonnegative integers $m_1, \ldots, m_p$ such that

$$M \cong D_1^{m_1} \oplus D_2^{m_2} \oplus \cdots \oplus D_{p-1}^{m_{p-1}} \oplus D_p^{m_p}$$

where each $D_i = k[x]/(x^i)$. Note that each D_i has k-basis $\{1, x, \ldots, x^{i-1}\}$, so the representation matrix of x on D_i is

$$\left[x\right]_{D_i} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

- These correspond to what we will call Jordan blocks. We will denote them as J_i.

Jordan Blocks

In general, J_i is the $i \times i$ matrix consisting of $1's$ on the subdiagonal and $0's$ everywhere else.

$$D_1 \longleftrightarrow J_1 = \begin{bmatrix} 0 \end{bmatrix}$$
$$D_2 \longleftrightarrow J_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$
$$D_3 \longleftrightarrow J_3 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$
$$D_4 \longleftrightarrow J_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

and so on. J_i is the representation matrix of x on D_i with respect to the k-basis $\{1, x, \ldots, x^{i-1}\}$.

Question

Question 2

Can we determine the decomposition of $M \otimes_k N$ over $R = k[u_\lambda]$ if we know the decompositions of M and N (both A-modules) over R?

Clebsch-Gordan Problem

- Let M, N be finitely generated A-modules. Then there exist nonnegative integers $m_1, \dots, m_p, n_1, \dots, n_p$ such that we have the following decompositions:

$$M \cong D_1^{m_1} \oplus D_2^{m_2} \oplus \cdots \oplus D_{p-1}^{m_{p-1}} \oplus D_p^{m_p}$$
$$N \cong D_1^{n_1} \oplus D_2^{n_2} \oplus \cdots \oplus D_{p-1}^{n_{p-1}} \oplus D_p^{n_p}.$$

- $M \otimes_k N$ is also an A-module, so we have

$$M \otimes_k N \cong D_1^{\ell_1} \oplus D_2^{\ell_2} \oplus \cdots \oplus D_{p-1}^{\ell_{p-1}} \oplus D_p^{\ell_p}.$$

Follow-up to Question 2

Can we write an explicit formula for each ℓ_i? In particular, what does $D_i \otimes_k D_j$ decompose as for some arbitrary $i, j \leq p$?

What we need

- Let M, N be finitely generated A-modules.
- We need to understand how u_λ acts on $M \otimes_k N$.
- Better question: How does each x_i act on $M \otimes_k N$?

Group Algebra Structure of A

- Let $G = \langle g_1 \rangle x \cdots x \langle g_n \rangle$ be a multiplicative group where each $\langle g_i \rangle$ is a cyclic group of order p.
- Consider the group algebra kG and let $\psi_1 : A \to kG$ be defined by $\psi_1(x_i) = g_i - 1$.
- Moreover, let $\psi_2 : kG \to A$ be defined by $\psi_2(g_i) = x_i + 1$.
- Note that ψ_1 and ψ_2 are inverses of each other, so we have that $kG \cong A$.

Why do we need kG?

- Since kG is a group algebra, we have the following diagonal map:

$$\Delta : kG \to kG \otimes_k kG$$
$$g_i \mapsto g_i \otimes g_i$$

- Since $kG \cong A$, we have the following commutative diagram:

$$\begin{array}{ccc} kG & \xrightarrow{\Delta} & kG \otimes_k kG \\ \downarrow & & \downarrow \\ A & \xrightarrow{\Delta_A} & A \otimes_k A \end{array}$$

- We need to identify what $\Delta_A(x_i)$ is.

Diagonal Map for A

- Note that $\psi_2(g_i - 1) = x_i$ since $\psi_2(g_i) = x_i + 1$.
- By the previous commutative diagram, we have that

$$\Delta_A \circ \psi_2(g_i - 1) = (\psi_2 \otimes_k \psi_2) \circ \Delta(g_i - 1)$$

which gives us

$$\Delta_A(x_i) = 1 \otimes x_i + x_i \otimes 1 + x_i \otimes x_i.$$

- Note that

$$\Delta_A(x) = 1 \otimes x + x \otimes 1 + x \otimes x$$

when we are dealing with one variable.

Using Representation Matrices

- Due to our definition for Δ_A, we determined that

$$[x]_{D_i \otimes_k D_j} = I_i \bigotimes [x]_{D_j} + [x]_{D_i} \bigotimes I_j + [x]_{D_i} \bigotimes [x]_{D_j}$$

  where $\bigotimes$ denotes the Kronecker product.

- Let $M = D_i \otimes_k D_j$ for some $i, j < p$ and recall that J_i is the representation matrix of x on D_i.

- Thus, we can derive the following formula:

$$\left[x\right]_M = I_i \bigotimes J_j + J_i \bigotimes I_j + J_i \bigotimes J_j = \begin{bmatrix} J_j & & & \mathbf{0} \\ I_j + J_j & \ddots & & \\ \mathbf{0} & \ddots & \ddots & \\ & & I_j + J_j & J_j \end{bmatrix}$$

- We will use this formula to help us with our decompositions.

Example 3

- Consider $M = D_2 \otimes_k D_2$ where $p > 2$. Then we have that

$$[x]_M = I_2 \bigotimes J_2 + J_2 \bigotimes I_2 + J_2 \bigotimes J_2 = \begin{bmatrix} J_2 & 0 \\ I_2 + J_2 & J_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

- Notice that our only eigenvalue of this matrix is 0, so we have that $\dim Null([x]_M - 0I) = \dim Null([x]_M) = 2$ since $rank([x]_M) = 2$.
- That means that this matrix's Jordan Normal form will be comprised of two Jordan blocks.
- Thus, by finding the biggest Jordan block, that will give us the other Jordan block for free.

Same Example

- Note that

$$[x]_M^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad [x]_M^3 = 0_{4x4}.$$

- Thus, we can see that the minimal polynomial of $[x]_M$ is $p(z) = z^3$.
- This means that the biggest Jordan block of our matrix is J_3. It immediately follows that our other Jordan block is J_1. So we have that the Jordan Normal form of $[x]_M$ is

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right].$$

- Therefore, we get that $D_2 \otimes_k D_2 \cong D_3 \oplus D_1$.

Results from Clebsh-Gordan Problem

- $D_i \otimes_k D_1 \cong D_i$ for any $1 \leq i \leq p$
- $D_i \otimes_k D_2 \cong D_{i+1} \oplus D_{i-1}$ for any $2 \leq i \leq p-1$
- $D_p \otimes_k D_j \cong D_p^j$ for any $1 \leq j \leq p$

free $\otimes_k$ anything = free!!

Known Theorem

$D_p \otimes_k D_j \cong D_p^j$ for any $j = 1, \ldots, p$

Proof: It is enough to show that the $rank([x]_{D_p \otimes_k D_j}) = (p-1)j$. Note that

$$[x]_{D_p \otimes_k D_j} = \begin{bmatrix} J_j & & & 0 \\ I_j + J_j & J_j & & \\ 0 & \ddots & \ddots & \\ & & I_j + J_j & J_j \end{bmatrix},$$

which is a $p \times p$ square matrix with $j \times j$ square matrix entries.

Continued Proof

Consider the $(p-1)j \times (p-1)j$ submatrix

$$\begin{bmatrix} I_j + J_j & J_j & & \mathbf{0} \\ & \ddots & \ddots & \\ & & \ddots & J_j \\ \mathbf{0} & & & I_j + J_j \end{bmatrix}.$$

I claim that this matrix has full rank. Consider the submatrix

$$\begin{bmatrix} I_j + J_j & J_j \end{bmatrix}.$$

Note that using a series of elementary row operations, we can get the matrix

$$\begin{bmatrix} I_j & \sum_{r=1}^{j-1} (-1)^{r-1} J_j^r \end{bmatrix}.$$

Let $J_j' = \sum_{r=1}^{j-1} (-1)^{r-1} J_j^r$. So we have that

Continued Proof pt.2

$$\begin{bmatrix} I_j + J_j & J_j & & 0 \\ & \ddots & \ddots & \\ & & \ddots & J_j \\ 0 & & & I_j + J_j \end{bmatrix} \rightarrow \begin{bmatrix} I_j & J'_j & & 0 \\ & \ddots & \ddots & \\ & & \ddots & J'_j \\ 0 & & & I_j \end{bmatrix}$$

which is an upper triangular matrix with $1's$ on the diagonal. Thus, by using cofacter expanson on the diagonal entries, we can see that it has determinant 1, which implies it has full rank. Therefore, the rank of $[x]_{D_p \otimes_k D_j}$ is $(p-1)j$ which implies that $D_p \otimes_k D_j$ is a free R-module.

Example 4

- Consider $M = D_3 \otimes_k D_3$ where $p = 3$. Then we have

$$[x]_M = \begin{bmatrix} J_3 & 0 & 0 \\ I_3 + J_3 & J_3 & 0 \\ 0 & I_3 + J_3 & J_3 \end{bmatrix} = \left[\begin{array}{ccc|ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \end{array}\right].$$

Same Example

- Consider the following submatrix:

$$\begin{bmatrix} I_3 + J_3 & J_3 \\ 0 & I_3 + J_3 \end{bmatrix} = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right].$$

- Perform the following elementary row operations in this order: $R_2 - R_1$, $R_3 - R_2$, $R_5 - R_4$, $R_6 - R_5$. Then we get

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right].$$

Future Goals

Goal 1: Further cases of the Clebsch-Gordan problem

Can we figure out more cases for $D_i \otimes_k D_j$?

Goal 2: Using natural Hopf Algebra structure

A is also considered to be a Hopf Algebra with a natural diagonal map $\Delta(x_i) = 1 \otimes_k x_i + x_i \otimes_k 1$. How does using this map affect the decomposition of a module? In particular, how does $D_i \otimes_k D_j$ decompose using this diagonal map?

Goal 3: Investigating $k[u_\lambda]$-modules over hypersurfaces with $char(k) = 0$

Let $\tilde{u_\lambda} = \lambda_1 x_1^p + \cdots + \lambda_n x_n^p \in k[x_1, \ldots, x_n]$ and consider the hypersurface $H = \frac{k[x_1,\ldots,x_n]}{(\tilde{u_\lambda})}$. Consider an A-module M. Does M have the "same behavior" over H that it does over $k[u_\lambda]$?

First Example of Goal 3

- Let $n = 2$ and $p = 2$. Then $A = \frac{k[x_1,x_2]}{(x_1^2,x_2^2)}$ and $H = \frac{k[x_1,x_2]}{(\lambda_1 x_1^2 + \lambda_2 x_2^2)}$.
- Consider the ideal $M = (x_1)$ of A from example 1. Recall that M is free over $k[u_\lambda]$ if and only if $\lambda_2 \neq 0$.
- Note that x_1 is a regular element in H if and only if $\lambda_2 \neq 0$. In this case, $M \cong H/(x_1)$ as an H-module.
- Thus, we have the following free resolution:

$$0 \to H \xrightarrow{[x_1]} H \to M \to 0.$$

- Therefore, $pd_H(M) = 1$ if and only if $\lambda_2 \neq 0$.

Second Example of Goal 3

- Let $n = 2$ and $p = 2$. Then $A = \frac{k[x_1,x_2]}{(x_1^2,x_2^2)}$ and $H = \frac{k[x_1,x_2]}{(\lambda_1 x_1^2+\lambda_2 x_2^2)}$.
- A has k-basis $\{1, x_1, x_2, x_1x_2\}$, so

$$\left[u_\lambda\right]_A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \lambda_1 & 0 & 0 & 0 \\ \lambda_2 & 0 & 0 & 0 \\ 0 & \lambda_2 & \lambda_1 & 0 \end{bmatrix}.$$

- A is free over $k[u_\lambda] \Leftrightarrow rank\left(\left[u_\lambda\right]_A\right) = 2 \Leftrightarrow$ one of $\lambda_1^2, \lambda_1\lambda_2, \lambda_2^2$ is nonzero.
- Now assume $char(k) \neq p$. Are those equations the same over H?

References

Bergh, Petter A; Erdmann, Karin; Jorgensen, David. *Modules of Constant Jordan Type over Quantum Complete Intersections.* DOCUMENTA MATHEMATICA, vol. Vol 25, Oct. 2020, pp. 1541-1570 Pages. DOI.org (Datacite), https://doi.org/10.25537/DM.2020V25.1541-1570.

Carlson, Jon F. *The Varieties and the Cohomology Ring of a Module,* Journal of Algebra, vol. 85, no. 1, Nov. 1983, pp. 104–43. DOI.org (Crossref), https://doi.org/10.1016/0021-8693(83)90121-7.

Darpö, Erik and Herschend, Martin *On the representation ring of the polynomial algebra over a perfect field,* Mathematische Zeirschrift vol. 265, no. 3, July 2010, pp. 601-15. DOI.org (Crossref) https://doi.org/10.1007/s00209-009-0532-9

Thank You!