The homotopy Lie algebra of a Tor-independent tensor product

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SETUP (R, m, k) LOCAL COMM, $I_1, I_2 \subseteq m$ IDEALS

$S_1 = R/I_1$, $S_2 = R/I_2$ $\quad S = S_1 \otimes_R S_2 = \frac{R}{I_1 + I_2}$

QUESTION: HOW DO THE HOMOLOGICAL PROPERTIES OF R, S_1, S_2, S RELATE?

IN A '75 PAPER AVRAMOV STUDIED THE RELATION BETWEEN THE TOR ALGEBRAS OF THESE RINGS.

THM (TATE, GULLIKSEN, SCHOELLER)

THERE IS A $\underbrace{\text{MINIMAL DGA RESOLUTION}}_{F}$ OF k /R

DEF: $\mathrm{Tor}^R(k,k) := F \otimes_R k$

THM (AVRAMOV '75):

IF R REGULAR, $I_1, I_2 \subseteq m^2$ AND $I_1 I_2 = I_1 \cap I_2$, THEN

$$\mathrm{Tor}^S(k,k) \cong \mathrm{Tor}^{S_1}(k,k) \otimes_{\mathrm{Tor}^R(k,k)} \mathrm{Tor}^{S_2}(k,k)$$

DEF: $\varphi: R \twoheadrightarrow S$ IT INDUCES A MAP $F_R \to F_S$

$\operatorname{Tor}^{\varphi}(k,k): \operatorname{Tor}^R(k,k) \to \operatorname{Tor}^S(k,k)$

φ IS <u>SMALL</u> IF $\operatorname{Tor}^{\varphi}(k,k)$ IS INJECTIVE

THM (AVRAMOV '78):

$$\begin{array}{ccc} R & \xrightarrow{\varphi_1} & S_1 \\ {\scriptstyle \varphi_2}\downarrow & & \downarrow \\ S_2 & \longrightarrow & S \end{array}$$

IF ONE OF THE φ_i IS SMALL AND $\operatorname{Tor}^R_{>0}(S_1, S_2) = 0$, THEN

$$\operatorname{Tor}^S(k,k) \cong \operatorname{Tor}^{S_1}(k,k) \otimes_{\operatorname{Tor}^R(k,k)} \operatorname{Tor}^{S_2}(k,k)$$

OBSERVATION: $\cdot$ $R \to R/I$ $\quad$ R REGULAR, $I \subseteq m^2$

THEN $R \to R/I$ SMALL

$\cdot$ R REGULAR $\quad \operatorname{Tor}^R_{>0}(S_1, S_2) = 0 \iff I_1 I_2 = I_1 \cap I_2$

OUR WORK: WE INVESTIGATED HOW THE HOMOTOPY LIE ALGEBRAS OF $\underbrace{\text{THESE}}_{R,\, S_1,\, S_2,\, S}$ RINGS RELATE

CONSTRUCTION: $F \xrightarrow{\simeq} k$ $\quad$ F MINIMAL DGA RES OF k

$\operatorname{Der}_R(F,F) = \{ \theta \in \operatorname{Hom}_R(F,F) \mid \theta$ SATISFIES THE LEIBNIZ RULE $\}$

NOTICE: $\mathrm{Der}_R(F,F)$ IS A DG LIE ALGEBRA

$[\ ,\ ]$: GRADED COMMUTATOR

DEF: THE HOMOTOPY LIE ALG $\pi(R) = H(\mathrm{Der}_R(F,F))$

DEF: $\varphi : R \to S$ IS ALMOST SMALL IF

$\ker \mathrm{Tor}^{\varphi}(k,k)$ IS GENERATED IN DEG 1

EX: $\varphi : R \twoheadrightarrow S$ R REG $\Rightarrow$ φ ALMOST SMALL

THM (FGSPP)

$$\begin{tikzcd} R \arrow[r, "\varphi_1"] \arrow[d, "\varphi_2"'] & S_1 \arrow[d] \\ S_2 \arrow[r] & S \end{tikzcd}$$

IF ONE OF THE φ_i IS ALMOST SMALL AND $\mathrm{Tor}^R_{>0}(S_1, S_2) = 0$, THEN

$$\pi(S) \cong \pi(S_1) \underset{\pi(R)}{\times} \pi(S_2)$$

MINIMAL MODELS

1) A SEMIFREE EXTENSION OF R IS A DGA $R[X]$ OBTAINED BY ADDING EXTERIOR VARIABLES IN ODD DEG AND POLYNOMIAL VARIABLES IN EVEN DEG

2) $R[X]$ IS A MINIMAL SEMIFREE EXT $\partial(k \otimes_R R[X]) \subseteq (X)^2$

3) $\varphi : R \twoheadrightarrow S$ $R[X]$ IS A MINIMAL MODEL FOR φ IF
- $R[X]$ IS A MINIMAL SF EXT
- $R[X] \xrightarrow{\simeq} S$

GULLIKSEN MINIMALITY

LET $\quad Q \xrightarrow{\varrho} R \xrightarrow{\varphi} S$

$Q[W]$ MIN MOD FOR ϱ

$R[X]$ MIN MOD FOR φ

USING LIFTING PROPERTIES $\quad Q[X, W] \xrightarrow[\simeq]{} S$

DEF: • φ IS ϱ-GULLIKSEN MINIMAL IF $Q[X, W]$ SATISFIES

$$\begin{array}{cc} \left(\cdots \; 1 \; {}^{\in \mathfrak{m}_Q} \; \cdots \right) & Y \\ X & \\ \text{EVEN} & \end{array}$$

• φ IS GULLIKSEN MINIMAL IF

$\acute{\varphi}$ IS ϱ-GULLIKSEN MINIMAL WITH RESPECT TO A COHEN PRESENTATION $\varrho : Q \to R$

LEMMA (GULLIKSEN)

IF $R[X]$ IS A GULLIKSEN MINIMAL SF EXT OR CHAR k = 0

THEN $\exists\, X' \subseteq X$ S/T $\quad R[X'] \xrightarrow[\text{QUISM}]{} R[X]$ AND

$R[X']$ IS A MINIMAL SF EXT.

FACT (AVRAMOV-IYENGAR, BRIGGS)

φ ALMOST SMALL $\Rightarrow$ φ GULLIKSEN MINIMAL

THM (FGSPD):
$$\begin{tikzcd} R \arrow[r, "\varphi_1"] \arrow[d, "\varphi_2"'] & S_1 \arrow[d] \\ S_2 \arrow[r] & S \end{tikzcd}$$

IF φ_1 AND φ_2 ARE GULLIKSEN MINIMAL AND $\mathrm{Tor}^R_{>0}(S_1, S_2) = 0$,

THEN $\pi(S) \cong \pi(S_1) \times_{\pi(R)} \pi(S_2)$

KEY INGREDIENT: LEMMA OF GULLIKSEN

THM (FGSPD): IF CHAR $k = 0$ AND $\mathrm{Tor}^R_{>0}(S_1, S_2) = 0$

THEN $\pi(S) \cong \pi(S_1) \times_{\pi(R)} \pi(S_2)$

APPLICATIONS

TOR ALGEBRAS

RECALL: $\mathfrak{g}$ LIE ALG

$$U\mathfrak{g} = \frac{T\mathfrak{g}}{(x \otimes y - (-1)^{|x||y|} y \otimes x - [x,y] \ , \ x, y \in \mathfrak{g})}$$

FACT: $\mathrm{Hom}_k(U\pi(R), k) \cong \mathrm{Tor}^R(k, k)$

FACT: • U PRESERVES PULLBACKS

• $\mathrm{Hom}_k(-, k)$ SENDS PULLBACKS TO PUSHOUTS

COROLLARY (FGJPP):

$$\begin{tikzcd} R \arrow[r, "\varphi_1"] \arrow[d, "\varphi_2"'] & S_1 \arrow[d] \\ S_2 \arrow[r] & S \end{tikzcd}$$

WITH $\mathrm{Tor}^R_{>0}(S_1, S_2) = 0$

IF ONE OF THE FOLLOWING HYPOTHESES HOLDS

1) φ_i IS ALMOST SMALL FOR SOME i

2) φ_1, φ_2 ARE GULLIKSEN MINIMAL

3) CHAR $K = 0$

THEN $\mathrm{Tor}^S(k,k) \cong \mathrm{Tor}^{S_1}(k,k) \underset{\mathrm{Tor}^R(k,k)}{\otimes} \mathrm{Tor}^{S_2}(k,k)$

GOLODNESS

RECALL: $\varphi : R \to S$ $\quad R[X] \to S$ MIN MODEL

THE FIBER OF φ IS $F^{\varphi} = R[X] \underset{R}{\otimes} k$

THM (FGJPP):

$$\begin{tikzcd} R \arrow[r, "\varphi_1"] \arrow[d, "\varphi_2"'] \arrow[dr, "\varphi"] & S_1 \arrow[d] \\ S_2 \arrow[r] & S \end{tikzcd}$$

$\mathrm{Tor}^R_{>0}(S_1, S_2) = 0$

THEN $\pi(F^{\varphi}) \cong \pi(F^{\varphi_1}) \times \pi(F^{\varphi_2})$

AVRAMOV: φ IS GOLOD $\iff$ $\pi(F^{\varphi})$ IS FREE

COROLLARY: IF $\mathrm{Tor}^R_{\geq 6}(S_1,S_2)=0 \implies \varphi: R \to S$ IS NOT GOLOD

STABLE COHOMOLOGY

DEF: IF A IS A GRADED CONNECTED k-ALG

$\mathrm{DEPTH}\, A = \inf\{n \geq 0 \mid \mathrm{Ext}^n_A(k,A) \neq 0\}$

THM (AVRAMOV-VELICHE): IF $\mathrm{DEPTH}\, \underset{\overset{\parallel}{UA(R)}}{\mathrm{Ext}_R(k,k)} \geq 2$

THEN $\widehat{\mathrm{Ext}}_R(k,k)$ HAS A SIMPLE STRUCTURE

THM (FERRARO): IF R IS GORENSTEIN AND

$\mathrm{DEPTH}\, \mathrm{Ext}_R(k,k) \geq 2$ THEN

$$\widehat{\mathrm{Ext}}_R(k,k) \cong \mathrm{Ext}_R(k,k) \ltimes \Sigma^{1-\dim R}\, \mathrm{Ext}_R(k,k)^*$$

COROLLARY (FG SPD):

$$\begin{array}{ccc} R & \xrightarrow{\varphi_1} & S_1 \\ {\scriptstyle \varphi_2}\downarrow & \searrow^{\varphi} & \downarrow \\ S_2 & \longrightarrow & S \end{array}$$

IF φ_1, φ_2 MINIMAL COHEN PRES

S_1, S_2 SINGULAR

$\mathrm{Tor}^R_{>0}(S_1, S_2) = 0$

THEN $\quad$ DEPTH $\mathrm{Ext}_S(k,k) \geq 2$

PROOF: $\quad$ DEPTH $\mathrm{Ext}_S(k,k) = $ DEPTH $U\pi(F^\varphi)$

$= $ DEPTH $U\pi(F^{\varphi_1}) + $ DEPTH $U\pi(F^{\varphi_2})$

$= $ DEPTH $\mathrm{Ext}_{S_1}(k,k) + $ DEPTH $\mathrm{Ext}_{S_2}(k,k)$

$\geq 1 + 1 = 2 \qquad \boxed{}$

PROP (FGSPP, MOORE - JORGENSEN)

$\mathrm{Tor}^R_{>0}(S_1, S_2)$

φ IS GOR $\iff \varphi_1, \varphi_2$ GOR

POINCARÉ SERIES

COROLLARY (FGSPP):

$$\begin{array}{ccc} R & \xrightarrow{\varphi_1} & S_1 \\ {\scriptstyle \varphi_2}\downarrow & & \downarrow \\ S_2 & \longrightarrow & S \end{array} \qquad \mathrm{Tor}^R_{>0}(S_1, S_2) = 0$$

IF AT LEAST ONE OF THE φ_i IS ALMOST SMALL,

THEN $\quad P^S_k(t) = \dfrac{P^{S_1}_k(t)\, P^{S_2}_k(t)}{P^R_k(t)}$

PROOF: $\varphi: R \to S$ ALMOST SMALL

$\iff \pi^{\geq 2}(S) \to \pi^{\geq 2}(R)$ IS SURJ

BY OUR THM $\pi^{\geq 2}(S) = \pi^{\geq 2}(S_1) \times_{\pi^{\geq 2}(R)} \pi^{\geq 2}(S_2)$

$$0 \to \pi^{\geq 2}(S) \to \pi^{\geq 2}(S_1) \times \pi^{\geq 2}(S_2) \to \underset{\text{ALMOST SMALL}}{\pi^{\geq 2}(R)} \to 0$$

$\dim \pi^i(S) = \dim \pi^i(S_1) + \dim \pi^i(S_2) - \dim \pi^i(R)$

$i \geq 2$

CHECK $\mathrm{Tor}^R_{>0}(S_1, S_2) = 0$ GIVES YOU THE EQUALITY FOR $i = 1$

FACT: $$P^R_k(t) = \prod_{i=1}^{\infty} \frac{(1+t^{2i-1})^{\dim \pi^{2i-1}(R)}}{(1-t^{2i})^{\dim \pi^{2i}(R)}}$$ $\blacktriangle$

ANDRÉ-QUILLEN COHOMOLOGY

THM (QUILLEN): IF CHAR $k = 0$

$\varphi: R \to S$, THEN

$\pi(F^\varphi) \cong D(S|R; k)$

THM (FGJPP):

$$\begin{tikzcd}
Q \arrow[dr, "\rho"] \arrow[drr, bend left, "\sigma_1"] \arrow[ddr, bend right, "\sigma_2"'] & & \\
& R \arrow[r, "\varphi_1"] \arrow[d, "\varphi_2"'] \arrow[dr, "\varphi"] & S_1 \arrow[d] \\
& S \arrow[r] & S
\end{tikzcd}$$

$\sigma := \varphi\rho$

IF CHAR $K=0$ (OR IF φ_1, φ_2 ARE ρ-GULLIKSEN MINIMAL) AND $\operatorname{Tor}^R_{>0}(S_1, S_2) = 0$, THEN

$$\pi(F^{\sigma}) \cong \pi(F^{\sigma_1}) \times_{\pi(F^{\rho})} \pi(F^{\sigma_2})$$

COROLLARY (FGJPP) IF CHAR $K=0$ AND $\operatorname{Tor}^R_{>0}(S_1, S_2) = 0$ THEN

$$D(S|Q;K) \cong D(S_1|Q;K) \times_{D(R|Q;K)} D(S_2|Q;K)$$

THM (QUILLEN): IF CHAR $K=0$ AND A, B ARE LOCAL NOETH K-ALG WITH RESIDUE FIELD K AND ESSENTIALLY OF FINITE TYPE, THEN

$$D(K|A \otimes_K B; K) \cong D(K|A;K) \times D(K|B;K)$$