"Some criteria for detecting large, small, and Golod homomorphisms"

Joint work with Ryo Takahashi

Throughout all rings are comm Noeth local.

Avramov 1978: Small homomorphisms

Levin 1979: Large homomorphisms

Let $f: R \longrightarrow S$ a surjective local hom

Then f induces $f_i : \operatorname{Tor}_i^R(k,k) \longrightarrow \operatorname{Tor}_i^S(k,k)$

- f is small if f_i is injective $\forall i$
- f is large if f_i is surjective $\forall i$
- f is Golod if in the following diagram:

$$\begin{tikzcd}
\operatorname{Tor}_i^R(k,k) \arrow[r, "f_i"] \arrow[d, "\delta_i"] & \operatorname{Tor}_i^S(k,k) \arrow[d, "\cong"] \\
\operatorname{Tor}_{i-1}^R(m_S,k) \arrow[r, "\gamma_i"] & \operatorname{Tor}_{i-1}^S(m_S,k)
\end{tikzcd}$$

$0 \to m_S \to S \to k \to 0$

$\forall i \geq 1$

both δ and γ are injective.

In particular any Golod hom. is small.

Thm [Levin]: TFAE

1. $f: R \longrightarrow S$ is large
2. $P_k^R = P_S^R \cdot P_k^S$
3. $P_M^R = P_S^R \cdot P_M^S$ for every f.g. S-module.

4. $\mathrm{Tor}_i^R(S,k) \longrightarrow \mathrm{Tor}_i^R(k,k)$ is injective for all i.

Necessary Conditions: $f: R \twoheadrightarrow S = \frac{R}{I}$

- f is small only if $I \subset m_R^2$.
- f is large only if $I \cap m_R^2 = I m_R$.

Examples: Let $I \cap m_R^2 = I m_R$. In either of the following $R \longrightarrow \frac{R}{I}$ is large.

1. $\mathrm{pd}_R I < \infty$
2. $(0 :_R I) = m_R$
3. $\frac{R}{I}$ is CI
4. $m = I \oplus J$

Notations: $K(I)$ = koszul complex of R w.r.t. a minimal generators of I.

$H_i(I) = H_i(K(I))$

$H_i(R) = H_i(K(m_R))$

Large Homomorphisms over CI:

Let R be a CI local ring and $I \cap m_R^2 = I m_R$. TFAE

1. $R \longrightarrow \frac{R}{I}$ is large
2. $\frac{R}{I}$ is CI

3. $H_1(I)\otimes k \longrightarrow H_1(R)$ is injective.

3'. $\mathrm{Tor}_2^R(\frac{R}{I},k) \longrightarrow \mathrm{Tor}_2^R(k,k)$ is injective.

4. $H_2(R) \to H_2(\frac{R}{I})$ is surjective.

4'. $\mathrm{Tor}_3^R(k,k) \longrightarrow \mathrm{Tor}_3^{R/I}(k,k)$ is surjective.

Example: $R = \dfrac{k[x,y,z]}{(x^2,y^2,z^2)}$

$I = (x+y+z)$ $\Rightarrow$ $\dfrac{R}{I} \cong \dfrac{k[y,z]}{(y,z)^2}$ Not CI

char $k \neq 2$ so $R \to \frac{R}{I}$ is not large.

Over CI ring R: $R\langle X_i\rangle \xrightarrow{\simeq} k$ Acyclic closure of k over R.

$X_i = 0$ for $i \geqslant 3$

Remarks: Let $f: R \to S$ be a local + surjective

1. If $\mathrm{Tor}_{i \gg 0}^R(k,k) \longrightarrow \mathrm{Tor}_{i\gg 0}^S(k,k)$ then f is large.

2. If $\mathrm{Tor}_{i\gg 0}^R(S,k) \longrightarrow \mathrm{Tor}_{i\gg 0}^R(k,k)$ is injective

not necessarily f is large.

one can take $\mathrm{pd}\, I < \infty$, $I \subset m_R^2$.

3. We don't know if $\mathrm{pd}_R I = \infty$ and $\mathrm{Tor}_{i\gg 0}^R(S,k) \to \mathrm{Tor}_{i\gg 0}^R(k,k)$ injective.

4. If $\mathrm{Tor}_{i\gg 0}^R(S,k) \longrightarrow \mathrm{Tor}_{i\gg 0}^R(k,k)$ injective $+ I \cap m_R^2 = I m_R$

and $H_1(I) \to H_1(R)$ is non-zero then f is large.

Thm: Let (R, m, k) be local and $I \neq 0$ with $I \cap m^2 = Im$.

TFAE:

(i) The maps $\mathrm{Tor}_i^R(mI, k) \longrightarrow \mathrm{Tor}_i^R(I, k)$ induced by $Im \hookrightarrow I$ is zero for all i.

(ii) The map $R \to \frac{R}{I}$ is large and $R \to \frac{R}{mI}$ is small

Moreover, under these equiv. conditions $R \to \frac{R}{mI}$ is Golod.

Koszul modules: Let be R graded ring and M a f.g. R-module. Then M is Koszul if M has linear free resolution over R. In other words $\mathrm{reg}_R M = 0$

R is called Koszul if k is a Koszul R-module.

Corollary: Let $I \cap m^2 = Im$. If $\frac{R}{I}$ is Koszul R-module then $R \to \frac{R}{I}$ is large and $R \to \frac{R}{mI}$ is Golod.

Recall: $R \to S$ is Golod iff

$$P_k^S = \frac{P_k^R}{1 - t(P_S^R - 1)}.$$

Corollary: If R is Koszul graded algebra and $R \to \frac{R}{I}$ is large, then $R \to \frac{R}{mI}$ is Golod and $\frac{R}{mI}$ is Koszul.

R is Golod if the map $Q \to \hat{R}$ is Golod hom.
where $Q \to \hat{R}$ is a min. Cohen factorization.

Prop. Let $f: R \to S$. If R is Golod and $H_i(R) \to H_i(S)$ is surjective for all $i \geq 1$ then f is large and also S is Golod.

Gupta: If R is Golod and $R \to S$ is large then S is Golod.

Example: $R = \dfrac{k[x,y,z]}{(x^2, xy, xz, y^2, z^2)}$

Let $I = (x)$ then $\dfrac{R}{(x)} = \dfrac{k[y,z]}{(y^2, z^2)}$ so $R \to \dfrac{R}{(x)}$ is large

but $R/(x)$ is not Golod. So R is not Golod too.

Minimal Intersections

Work under progress with L. Ferraro, D. Jorgensen,
N. Packauskas and J. Pollitz.

We say R is a minimal intersection if it fits in

$$\begin{tikzcd} Q \arrow[r, "\varphi_1"] \arrow[d, "\varphi_2"'] \arrow[dr, dashed, "\varphi"] & R_1 = \frac{Q}{I} \arrow[d, "\psi_2"] \\ R_2 = Q/J \arrow[r, "\psi_1"'] & R = \frac{Q}{I+J} \end{tikzcd} \quad \text{and } \operatorname{Tor}^Q_{i>0}(R_1, R_2) = 0$$

when Q is reg. it is equivalent to say $I \cap J = IJ$

Thm (Avramov 1978): If φ_1 or φ_2 is small then there is an isomorph. of Hopf algebras

$$\operatorname{Tor}^R(k,k) \cong \operatorname{Tor}^{R_1}(k,k) \otimes_{\operatorname{Tor}^Q(k,k)} \operatorname{Tor}^{R_2}(k,k)$$

Corollary: $P_k^R = \dfrac{P_k^{R_1} \cdot P_k^{R_2}}{P_k^Q}$.

We looked at homotopy Lie alg. of R, R_1, R_2 and Q

Main Thm: with the assumptions above (R is min. int.) then there is an isom. of graded Lie algebras

$$\pi(\varphi) \xrightarrow{\cong} \pi(\varphi_1) \oplus \pi(\varphi_2)$$

Consequences:

$$\begin{tikzcd} Q \arrow[r, "\varphi_1"] \arrow[d, "\varphi_2"'] \arrow[dr, dashed, "\varphi"] & R_1 = Q/I \arrow[d, "\psi_2"] \\ R_2 = Q/J \arrow[r, "\psi_1"'] & R = \frac{Q}{I+J} \end{tikzcd}$$

1. φ can not be Golod hom.

2. If φ_i is small then ψ_i is small.

3. φ_1 is large $\Longleftrightarrow$ ψ_1 is large and

$$P_k^R = \frac{P_k^{R_1} \cdot P_k^{R_2}}{P_k^Q}.$$

Proof: φ_1 large $\Rightarrow$ $P_k^Q = P_{R_1}^Q \cdot P_k^{R_1}$

ψ_1 large $\Rightarrow$ $P_k^{R_2} = P_R^{R_2} \cdot P_k^R$

$P_{R_1}^Q = P_R^{R_2}$ since $\mathrm{Tor}_i^Q(R_1, R_2) = 0 \quad \forall i > 0$

If Q reg. and R cm then

$$I_R^R = \frac{I_{R_1}^{R_1} \cdot I_{R_2}^{R_2}}{I_Q^Q}.$$