Bumpless pipe dreams encode Gröbner geometry of Schubert polynomials.

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A classical problem: Given four lines in complex 3-space in general position, how many lines meet all four given lines?

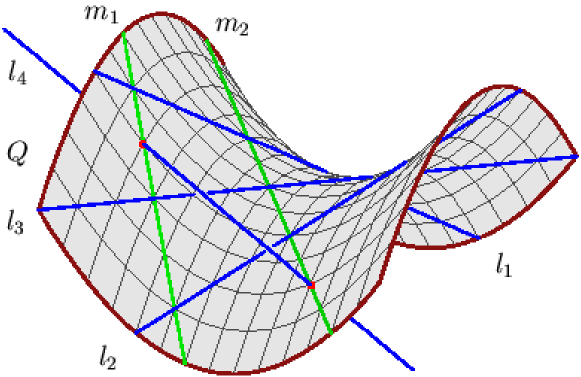

Image from Frank Sottile's online Schubert calculus notes.

Schubert varieties

A classical problem (alternatively phrased): Given four planes through the origin in complex projective 4-space, how many planes through the origin meet all four given planes elsewhere?

Schubert varieties

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Instead of studying Schubert varieties in $G(2,4)$, we could if we wanted study their preimages in the complete flag variety, which is a particularly nice quotient of the general linear group, and where many other of these enumerative geometry problems live.

The complete flag variety $\mathcal{F}\ell(\mathbb{C}^n)$

A *complete flag* is a chain

$$0 \subset V_1 \subset \cdots \subset V_n = \mathbb{C}^n$$

of complex vector spaces so that $\dim(V_i) = i$. You might take V_i to be the span of the first i columns of your favorite invertible matrix.

Note that $GL(\mathbb{C}^n)$ acts transitively on $\mathcal{F}\ell(\mathbb{C}^n)$, which allows us to identify $\mathcal{F}\ell(\mathbb{C}^n)$ with $GL(\mathbb{C}^n)/U$, where $U = \{$upper triangular matrices$\}$.

Bruhat decomposition and Schubert cells

Every matrix $M \in GL(\mathbb{C}^n)$ can be written $M = \ell wu$, where ℓ is a product of lower triangular matrices, w is a permutation matrix, and u is a product of upper triangular matrices ($\ell \in L, u \in U$). The decomposition $GL(\mathbb{C}^n) = \sqcup_{w \in S_n} LwU$ is called the *Bruhat decomposition* of $GL(\mathbb{C}^n)$.

The image of each LwU in $\mathcal{F}\ell(\mathbb{C}^n)$ is called a *Schubert cell*, and its closure is called a *Schubert variety*.

Schubert varieties in commutative algebra

- Classical determinantal rings (which are the homogeneous coordinate rings of open patches of Schubert varieties in Grassmannians) are normal Cohen–Macaulay domains. (Hochster and Eagon, 1971)
- Schubert subvarieties of Grassmannians are Cohen–Macaulay. The coordinate rings of Grassmannians are Gorenstein UFDs. (Hochster, 1973)
- Schubert varieties in $GL(\mathbb{C}^n)/U$ are arithmetically Cohen–Macaulay. (Musili and Seshadri, 1983)
- Ramanathan (1985), using Frobenius splitting, showed all Schubert varieties are arithmetically Cohen–Macaulay.

The cohomology ring of the complete flag variety $\mathcal{F}\ell(\mathbb{C}^n)$

Borel (1953) showed that the integral cohomology ring of the complete flag variety has the presentation

$$H^*(\mathcal{F}\ell(\mathbb{C}^n)) \cong \mathbb{Z}[x_1, \ldots, x_n]/I,$$

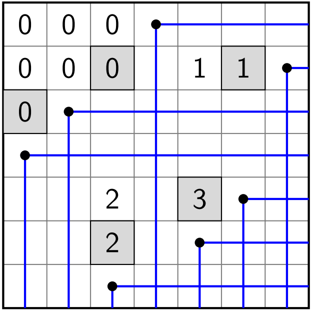
where I is the ideal generated by the nonconstant elementary symmetric polynomials.

What's more, Lascoux and Schützenberger (1982) showed that the elements of $\mathbb{Z}[x_1, \ldots, x_n]/I$ have combinatorially-natural representatives called *Schubert polynomials*. They also introduced *double Schubert polynomials*, which are refinements of Schubert polynomials that give the torus-equivariant cohomology.

Fulton's Matrix Schubert varieties

Each permutation $w \in S_n$ corresponds to a generalized determinantal variety called a *matrix Schubert variety*, introduced by Fulton (1992).

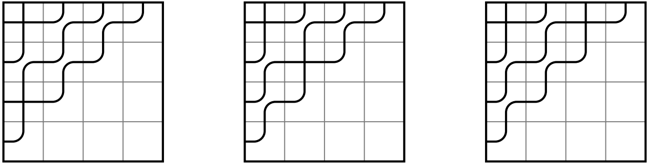
Example: Let $w = 4721653$.



The minors prescribed above are called the *Fulton generators* of the *Schubert determinantal ideal*. *Essential boxes* are shaded in grey.

Writing down double Schubert polynomials

Knutson and Miller (2005) showed that the *reduced pipe dreams* of a permutation w give the double Schubert polynomial of the corresponding Schubert variety X_w. For example, if $w = 2143$, then the reduced pipe dreams of w are

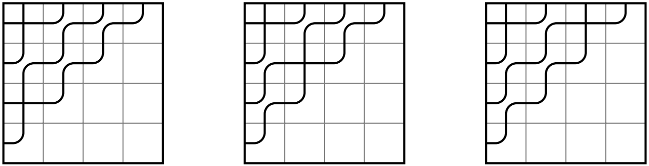


and the corresponding double Schubert polynomial as

$$\mathfrak{S}_w = (x_1 - y_1)(x_3 - y_1) + (x_1 - y_1)(x_2 - y_2) + (x_1 - y_1)(x_1 - y_3).$$

Their proof goes through the argument that the Fulton generators of the matrix Schubert variety of w are Gröbner with respect to any anti-diagonal term order.

Writing down double Schubert polynomials



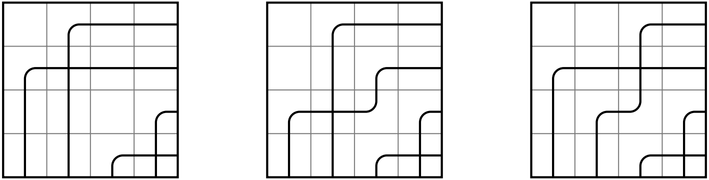
These anti-diagonal initial ideals are all reduced, and their prime components are indexed by the reduced pipe dreams.

In this example, $I_w = \left(x_{11}, \begin{vmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{vmatrix} \right)$, and

$$\mathrm{in}_<(I_w) = (x_{11}, x_{31}) \cap (x_{11}, x_{22}) \cap (x_{11}, x_{13}).$$

A coincidence in need of explanation

Bumpless pipe dreams, newer combinatorial objects growing from the work of Bergeron-Billey '93 and Fomin-Kirillov '96, can also be used to write down double Schubert polynomials (Lam–Lee–Shimozono '18, Lascoux '02+Weigandt '20). Again with $w = 2143$, the three reduced bumpless pipe dreams are



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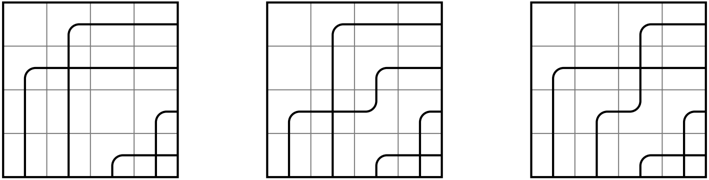

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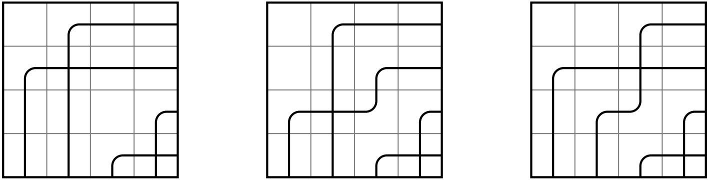
Diagonal degenerations of matrix Schubert varieties



In this example, $I_w = \left(x_{11}, \begin{vmatrix} 0 & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{vmatrix} \right)$, and

$$\mathrm{in}_<(I_w) = (x_{11}, x_{33}) \cap (x_{11}, x_{21}) \cap (x_{11}, x_{12}).$$

Main result

Theorem (K.–Weigandt)

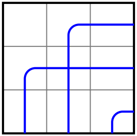
If $<$ is a diagonal term order, then the irreducible components of $\mathrm{in}_<(I_w)$, counted with scheme-theoretic multiplicity, naturally correspond to the bumpless pipe dreams of w.

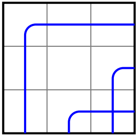
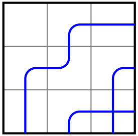
This result was conjectured by Hamaker, Pechenik, and Weigandt (2020), who showed the result in a special case, which extended the *vexillary* case given by Knutson, Miller, and Yong (2009).

Vexillary matrix Schubert varieties are also known as *one-sided ladder determinantal ideals*, under which name their Gröbner bases had been studied by Gonciulea and Miller (2000) and Gorla (2008).

An example: $w_1 = 213$, $w_2 = 132$.

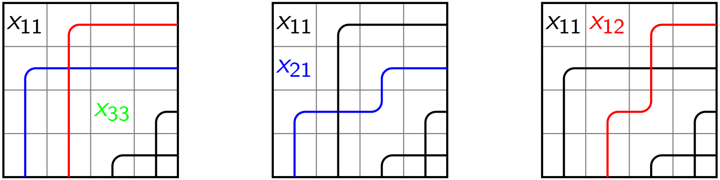
$I_{w_1} = (z_{11})$, $I_{w_2} = \left(\begin{vmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{vmatrix} \right)$. Set $J = I_{w_1} \cap I_{w_2}$. Then $\mathrm{in}_<(J) = (z_{11}^2 z_{22}) = (z_{11}^2) \cap (z_{22})$, so $\mathrm{mult}_{I_{\{(1,1)\}}}(\mathrm{in}_<(J)) = 2$, and $\mathrm{mult}_{I_{\{(2,2)\}}}(\mathrm{in} < (J)) = 1$.

$\mathrm{BPD}(w_1) = \Bigg\{$  $\Bigg\}$

$\mathrm{BPD}(w_2) = \Bigg\{$  ,  $\Bigg\}$.

Understanding droops as geometric vertex decomposition

Geometric vertex decomposition was introduced by Knutson–Miller–Yong (2009). With $w = 2143$, we will take x_{33} largest.



In this example, $I_w = \left(x_{11}, x_{21}x_{12}x_{33} + x_{12}x_{31}x_{23} - \begin{vmatrix} x_{21} & x_{22} \\ x_{31} & x_{32} \end{vmatrix} x_{13} \right)$

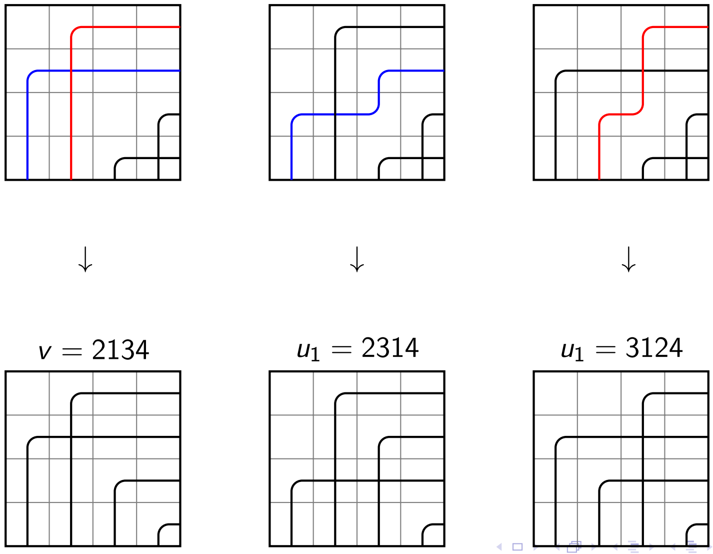
Then

$$\mathrm{in}_{x_{33}}(I_w) = C_{x_{33},I_w} \cap (N_{x_{33},I_w} + (x_{33})) = (x_{21}x_{12}, x_{11}) \cap (x_{11}, x_{33})$$

C_{x_{33},I_w} captures the primes of $\mathrm{in}_<(I_w)$ that *do not* contain x_{33}, and N_{x_{33},I_w} captures those that do.

A bijection on BPDs

$$\mathrm{in}_{x_{33}}(I_w) = C_{x_{33},I_w} \cap (N_{x_{33},I_w} + (x_{33})) = (I_{u_1} \cap I_{u_2}) \cap (I_v + (x_{33})).$$



Three recurrences

Fix a lower outside corner y, and take it to be lexicographically largest. $N_{y,I_w} = I_v$ is another Schubert determinantal ideal, and $C_{y,I_w} = \bigcap I_{u_i}$ is an intersection of Schubert determinantal ideals.

- The geometric vertex decomposition $\mathrm{in}_y I_w = C_{y,I_w} \cap (N_{y,I_w} + (y))$ divides the minimal primes of $\mathrm{in}_< I$ into those that don't contian y and those that do
- Drooping into y or "canceling" y spits the BPDs of w into the BPDs of C_{y,I_w} and those of N_{y,I_w}
- Lascoux's transition formula splits permutations "near" w into permutations corresponding to C_{y,I_w} and the one corresponding to N_{y,I_w}.

Applications to Cohen–Macaulayness

Theorem (K.–Rajchgot, 2021)

A geometric vertex decomposition like the above gives rise to an elementary G-biliaison.

Theorem (Hartshorne, 2007)

An elementary G-biliaison transfers to Cohen–Macaulay property.

Theorem (K.–Weigandt)

Let $w_1, \ldots, w_r$ be permutations of the same Coxeter length, and set $J = \bigcap I_{w_i}$. If Spec(R/J) and Spec$(R/N_{y,J})$ are Cohen–Macaulay, then Spec$(R/\mathrm{in}_y(J))$ and Spec$(R/C_{y,J})$ are Cohen–Macaulay as well. In particular, Spec$(R/\mathrm{in}_y(I_w))$ and Spec$(R/C_{y,I_w})$ are Cohen–Macaulay for all $w \in S_n$.

Note: Each C_{y,I_w} is the defining ideal of an *alternating sign matrix variety*.

What we still don't know

• A diagonal Gröbner basis from I_w, even in the radical case.
(Known in a special case classified by pattern avoidance, K. 2020)
• How higher multiplicities can manifest in the initial ideal
• When $\mathrm{in}_{<}(I_w)$ has embedded primes
• If there are any other formulas for double Schubert polynomials that make sense both combinatorially and algebro-geometrically
• Which ASMs are Cohen–Macaulay

Thank you!