Base Change Along the Frobenius Endomorphism And The Gorenstein Property

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Motivation

This talk is about the structure and properties of the category of complexes of finite injective dimension over commutative Noetherian rings.

Some well known examples.

- A local ring $(R, \mathfrak{m}, k)$ is regular if and only if k has finite injective dimension.
- A local ring $(R, \mathfrak{m}, k)$ that admits a finitely generated module of finite injective dimension is Cohen-Macaulay.
- A commutative Noetherian ring R admits a finitely generated module of finite injective and projective dimension if and only if R is Gorenstein.

Introduction

Setting and Notation:

$(R, \mathfrak{m}, k)$ a local ring.

An endomorphism $\varphi\colon R \to R$ is *contracting* if $\varphi^i(\mathfrak{m}) \subseteq \mathfrak{m}^2$ for some $i > 0$.

With φ contracting, we set R^φ to be R with the right module structure induced by φ.

$\mathsf{I}(R)$ is the subcategory of complexes of finite injective dimension.

$\mathsf{I}^{\mathrm{fg}}(R)$ is the subcategory of $\mathsf{I}(R)$ consisting of complexes with finitely generated homology.

$\mathsf{I}^{\mathrm{fl}}(R)$ is the subcategory of $\mathsf{I}^{\mathrm{fg}}(R)$ consisting of complexes with finitely length homology.

Examples of Contracting Endomorphisms

In characteristic $p > 0$, the Frobenius is contracting.

$R = k[x, y]/(x^3, y^3)$ with $\varphi(x) = y$ and $\varphi(y) = y^2$

There are interesting examples coming from semi-group rings.

Motivation

Theorem 1 (Falahola, Marley; 2018)

Let φ be a contracting endomorphism on a Cohen-Macaulay local ring R, and ω_R a canonical module. Then $\text{inj-dim}_R(\mathrm{R}^{\varphi} \otimes_R \omega_R) < \infty$ *if and only if R is Gorenstein.*

Question: Does the theorem hold if ω_R is a dualizing complex?

A: NO!

Recall: A complex D is a *dualizing complex* if the following hold:

(i) $\text{inj-dim}_R(D) < \infty$.

(ii) $H(D)$ is finitely generated.

(iii) The natural map $R \to \mathrm{RHom}_R(D, D)$ is a quasi-isomorphism.

Question (Falahola, Marley; 2018): Let φ be contracting, and D a dualizing complex. If $\text{inj-dim}_R(\mathrm{R}^\varphi \otimes^{\mathbf{L}}_R D) < \infty$ is R necessarily Gorenstein?

Main result

Theorem 2 (-)

Let $\varphi\colon R \to R$ be a contracting endomorphism. The following are equivalent.

(i) *R is Gorenstein.*

(ii) *There exists an R-complex $X \in \mathsf{I}^{\mathrm{fg}}(R)$ with $H(X) \neq 0$ such that,* $\operatorname{inj-dim}_R(\mathrm{R}^{\varphi} \otimes^{\boldsymbol{L}}_R X) < \infty$.

(iii) *For every $X \in \mathsf{I}^{\mathrm{fg}}(R)$ we have* $\operatorname{inj-dim}_R(\mathrm{R}^{\varphi} \otimes^{\boldsymbol{L}}_R X) < \infty$.

Outline of Proof

$(i) \implies (iii)$ is well known due to Foxby.

$(iii) \implies (ii)$ is easy. We just need to show that every local ring has a complex $X \in \mathsf{I}^{\mathrm{fg}}(R)$ with nonzero homology. The complex $K^R \otimes_R E(k)$ does the trick.

$(ii) \implies (i)$ is the new implication.

Outline of $(ii) \implies (i)$

Let $X \in \mathsf{I}^{\mathrm{fg}}(R)$ with $H(X) \neq 0$ s.t. $\text{inj-dim}_R(\mathrm{R}^{\varphi} \otimes^{\mathbf{L}}_R X) < \infty$.
We have a series of implications:

(a) $\implies$ there exists some $Y \in \mathsf{I}^{\mathrm{fl}}(R)$ with $H(Y) \neq 0$ s.t. $\text{inj-dim}_R(\mathrm{R}^{\varphi} \otimes^{\mathbf{L}}_R Y) < \infty$.

(b) $\implies$ for all $Y \in \mathsf{I}^{\mathrm{fl}}(R)$ we have $\text{inj-dim}_R(\mathrm{R}^{\varphi} \otimes^{\mathbf{L}}_R Y) < \infty$.

(c) $\implies$ for all $Y \in \mathsf{I}^{\mathrm{fl}}(R)$ and for all $i > 0$ we have $\text{inj-dim}_R(\mathrm{R}^{\varphi^{\mathrm{i}}} \otimes^{\mathbf{L}}_R Y) < \infty$.

(d) $\implies$ $\text{proj-dim}_R(Y) < \infty$.

(e) $\implies$ that R is Gorenstein.

Thick Subcategories

Definition 3

A non-empty subcategory $\mathcal{T}$ of $\mathsf{D}(R)$ is *thick* if it is full , closed under taking direct summands and for every exact triangle

$$X \to Y \to Z \to \Sigma X$$

if any two of X, Y, Z belong to $\mathcal{T}$, so does the third.

Example 4

The subcategories $\mathsf{I}^{\mathrm{fg}}(R)$, $\mathsf{I}^{\mathrm{fl}}(R)$ and $\mathsf{perf}(R)$ are all thick.

Definition 5

Given $X \in \mathsf{D}(R)$, the *thick subcategory generated by* X, denoted $\operatorname{Thick}_R(X)$, is the smallest thick subcategory that contains X. It is the intersection of all thick subcategories of $\mathsf{D}(R)$ containing X.

Example 6

Let R be a Noetherian ring. We always have $\operatorname{Thick}_R(R)$ are the perfect complexes. When $(R, \mathfrak{m}, k)$ is local we have $\operatorname{Thick}_R(k) = \mathsf{D}^{\mathrm{fl}}_{\mathrm{b}}(R)$.

Construction of $\mathrm{Thick}_R(X)$

We can construct $\mathrm{Thick}_R(X)$ as follows: Set

(i) $\mathrm{Thick}^0_R(X) = \{0\}$.

(ii) $\mathrm{Thick}^1_R(X)$ are the direct summands of finite direct sums of shifts of X.

(iii) For each $n \geq 2$, the objects of $\mathrm{Thick}^n_R(X)$ are direct summands of objects U such that U appears in an exact triangle

$$U' \to U \to U'' \to \Sigma U'$$

where $U'' \in \mathrm{Thick}^{n-1}_R(X)$ and $U' \in \mathrm{Thick}^1_R(X)$.

The subcategory $\mathrm{Thick}_R^n(X)$ is the *nth thickening* of X. Every thickening embeds in the next one.
We have a filtration:

$$\mathrm{Thick}_R^0(X) \subseteq \mathrm{Thick}_R^1(X) \subseteq \mathrm{Thick}_R^2(X) \subseteq ... \subseteq \bigcup_{n\geq 0} \mathrm{Thick}_R^n(X)$$

It turns out that

$$\mathrm{Thick}_R(X) = \bigcup_{n\geq 0} \mathrm{Thick}_R^n(X)$$

If $F\colon \mathsf{D}(R) \to \mathsf{D}(S)$ is an exact functor then

$$F(\operatorname{Thick}_R(X)) \subseteq \operatorname{Thick}_S(F(X)).$$

For any $X \in \mathsf{D}(R)$ and any perfect complex P, the complexes $P \otimes^{\mathbf{L}}_R X$ and $\operatorname{RHom}_R(P, X)$ are in $\operatorname{Thick}_R(X)$.

The *support* of an R-complex X is

$$\operatorname{Supp}_R(X) \coloneqq \{\mathfrak{p} \in \operatorname{Spec}(R) \mid H(X)_{\mathfrak{p}} \neq 0\}.$$

Support and Thick Subcategories

When $Y \in \operatorname{Thick}_R(X)$, we have

$$\operatorname{Supp}_R(Y) \subseteq \operatorname{Supp}_R(X).$$

The converse doesn't hold in general, but it does for perfect complexes.

Hopkins' and Neeman's Theorem

Theorem 7 (Hopkins, Neeman)

Let R be a commutative Noetherian ring. Given perfect R-complexes N and M, if $\operatorname{Supp}_R N \subseteq \operatorname{Supp}_R M$ then $N \in \operatorname{Thick}_R(M)$.

Proof.

"Left as an exercise to the reader." □

Hopkins-Neeman is very useful in many situations and often simplifies proofs.

For example, Let $(R, \mathfrak{m}, k)$ be a regular local ring. Then $k \in \mathrm{Thick}_R(X)$ for every $X \in \mathsf{D}^{\mathrm{fl}}_{\mathrm{b}}(R)$ when viewed in $\mathsf{D}(R)$.

However, in characteristic 0, it is not known if $k \in \mathrm{Thick}_R(M)$ in the module category when M has finite length.

Partial Analogue for $\mathsf{I}^{\mathrm{fl}}(R)$

Proposition 8

For all $X \in \mathsf{I}^{\mathrm{fl}}(R)$ *with* $H(X) \neq 0$*, one has* $\mathrm{Thick}_R(X) = \mathsf{I}^{\mathrm{fl}}(R)$.

Outline of proof.

It suffices to show that for all $Y \in \mathsf{I}^{\mathrm{fl}}(R)$ we have $Y \in \mathrm{Thick}_R(X)$. Take the Matlis dual $Y^\vee$, this is a perfect complex supported at $\{\mathfrak{m}\}$. By Hopkins-Neeman, $Y^\vee \in \mathrm{Thick}_R(X^\vee)$, $\implies Y^{\vee\vee} \in \mathrm{Thick}_R(X^{\vee\vee})$. $\square$

As a corollary we get the implication $(a) \implies (b)$.

Corollary 9

If there exists $X \in \mathsf{I}^{\mathrm{fl}}(R)$ *with* $H(X) \neq 0$ *s.t.* $\operatorname{inj-dim}_R(\mathrm{R}^{\varphi} \otimes_R^{\boldsymbol{L}} X) < \infty$*, then for all* $Y \in \mathsf{I}^{\mathrm{fl}}(R)$ *we have* $\operatorname{inj-dim}_R(\mathrm{R}^{\varphi} \otimes_R^{\boldsymbol{L}} Y) < \infty$*.*

Proof.

Since $Y \in \operatorname{Thick}_R(X)$, we have

$$\mathrm{R}^{\varphi^{\mathrm{i}}} \otimes_R^{\mathbf{L}} Y \in \operatorname{Thick}_R(\mathrm{R}^{\varphi^{\mathrm{i}}} \otimes_R^{\mathbf{L}} X) = \mathsf{I}^{\mathrm{fl}}(R)$$

$\square$

Outline of $(ii) \implies (i)$

Let $X \in \mathsf{I}^{\mathrm{fg}}(R)$ with $H(X) \neq 0$ s.t. $\operatorname{inj-dim}_R(\mathrm{R}^{\varphi} \otimes^{\mathbf{L}}_R X) < \infty$.
We have a series of implications:

(a) $\implies$ there exists some $Y \in \mathsf{I}^{\mathrm{fl}}(R)$ with $H(Y) \neq 0$ s.t. $\operatorname{inj-dim}_R(\mathrm{R}^{\varphi} \otimes^{\mathbf{L}}_R Y) < \infty$.

(b) $\implies$ for all $Y \in \mathsf{I}^{\mathrm{fl}}(R)$ we have $\operatorname{inj-dim}_R(\mathrm{R}^{\varphi} \otimes^{\mathbf{L}}_R Y) < \infty$.

(c) $\implies$ for all $Y \in \mathsf{I}^{\mathrm{fl}}(R)$ and for all $i > 0$ we have $\operatorname{inj-dim}_R(\mathrm{R}^{\varphi^{\mathrm{i}}} \otimes^{\mathbf{L}}_R Y) < \infty$.

(d) $\implies$ $\operatorname{proj-dim}_R(Y) < \infty$.

(e) $\implies$ that R is Gorenstein.

Loewy Length

Definition 10

Let $(R, \mathfrak{m}, k)$ be a local ring, X a complex of R modules. The *Loewy length* of X is defined to be

$$\ell\ell_R(X) \coloneqq \inf\{i \in \mathbb{N} \mid \mathfrak{m}^i \cdot X = 0\}$$

The *homotopical Loewy length* of X is defined to be

$$\ell\ell_{\mathsf{D}(R)}(X) \coloneqq \inf\{\ell\ell_R(V) \mid V \simeq X\}$$

Finiteness Property

Theorem 11 (Avramov, Iyengar, Miller)

Let $(R, \mathfrak{m}, k)$ be a local ring. Let K^R be the Koszul complex on a minimal generating set of $\mathfrak{m}$. For any complex X we have

$$\ell\ell_{\mathsf{D}(R)}(K^R \otimes^{\boldsymbol{L}}_R X) \leq \ell\ell_{\mathsf{D}(R)}(K^R) < \infty$$

Proposition 12

Let $\varphi\colon (R,\mathfrak{m},k) \to (S,\mathfrak{n},l)$ *be a local homomorphism such that* $\varphi(\mathfrak{m}) \subseteq \mathfrak{n}^c$ *where* $c = \ell\ell_{\mathsf{D}(S)}(K^S)$*. For any* $X \in \mathsf{D}^{\mathrm{fg}}_{+}(R)$ *we have* $\mathrm{Tor}_i^R(S,X) = 0$ *for all* $i \gg 0$ *if and only if* X *has finite projective dimension.*

Proof of Proposition.

The if part is clear. For the converse, note

$$\sup H(S \otimes^{\mathbf{L}}_R X) < \infty \implies \sup H(K^S \otimes^{\mathbf{L}}_R X) < \infty.$$

The complex $K^S \simeq H(K^S)$ in $\mathsf{D}(R)$. Since $H(K^S)$ has a k-vector space structure as an R complex, one gets by the Künneth formula

$$H(K^S \otimes^{\mathbf{L}}_R X) \cong H(K^S) \otimes_k H(k \otimes^{\mathbf{L}}_R X)$$

Since $H(K^S \otimes^{\mathbf{L}}_R X)$ is bounded, so is $H(k \otimes^{\mathbf{L}}_R X)$. Therefore $\operatorname{proj-dim}_R(X) < \infty$. □

Remark

If $\varphi\colon R \to R$ is a contracting endomorphism, then for i large enough $\varphi^i(\mathfrak{m}) \subseteq \mathfrak{m}^c$ where $c = \ell\ell_{\mathsf{D}(R)}(K^R)$. Hence the complex $\mathrm{R}^{\varphi^{\mathrm{i}}} \otimes^{\mathbf{L}}_R X$ has bounded homology for all $i \gg 0$ if and only if $\text{proj-dim}_R(X) < \infty$.

Outline of $(ii) \implies (i)$

Let $X \in \mathsf{I}^{\mathrm{fg}}(R)$ with $H(X) \neq 0$ s.t. $\text{inj-dim}_R(\mathrm{R}^{\varphi} \otimes^{\mathbf{L}}_R X) < \infty$.
We have a series of implications:

(a) $\implies$ there exists some $Y \in \mathsf{I}^{\mathrm{fl}}(R)$ with $H(Y) \neq 0$ s.t. $\text{inj-dim}_R(\mathrm{R}^{\varphi} \otimes^{\mathbf{L}}_R Y) < \infty$.

(b) $\implies$ for all $Y \in \mathsf{I}^{\mathrm{fl}}(R)$ we have $\text{inj-dim}_R(\mathrm{R}^{\varphi} \otimes^{\mathbf{L}}_R Y) < \infty$.

(c) $\implies$ for all $Y \in \mathsf{I}^{\mathrm{fl}}(R)$ and for all $i > 0$ we have $\text{inj-dim}_R(\mathrm{R}^{\varphi^{\mathrm{i}}} \otimes^{\mathbf{L}}_R Y) < \infty$.

(d) $\implies$ $\text{proj-dim}_R(Y) < \infty$.

(e) $\implies$ that R is Gorenstein.

Finite injective dimension is needed

Example 13

Let $R = k[x, y]/(x^3, y^3)$ with $\varphi(x) = y$ and $\varphi(y) = y^2$. Let X be the complex

$$\cdots \to R \xrightarrow{x^2} R \xrightarrow{x} R \xrightarrow{x^2} R \xrightarrow{x} R \to 0$$

it's easy to see that $X \simeq R/(x)$ and $\mathrm{R}^{\varphi} \otimes^{\mathbf{L}} X \simeq R/(y)$. However $\mathrm{R}^{\varphi^2} \otimes^{\mathbf{L}} X$ is not homologically bounded.

Questions?