Finding maximal Cohen Macaulay and reflexive modules

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Outline

- Motivation for this study.
- Some questions on existence of maximal Cohen Macaulay modules in mixed characteristic.
- Reflexivity in dimension one over non Gorenstein rings.

Definition 1

Let $(R, \mathfrak{m})$ be a local ring of dimension d. A non zero R-module M is a **maximal Cohen Macaulay module** (**small Cohen Macaulay module**) if it is finitely generated and every (equivalently some) system of parameters of R is a regular sequence on M .

Definition 2

Let R be a Noetherian ring and M a non zero R-module. Then M is reflexive if the natural map $M \to M^{**}$ is an isomorphism, where $M^* = Hom_R(M, R)$.

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How are they related ?

- Over Cohen Macaulay rings of $dim \leq 2$, Reflexive $\implies$ MCM.
- Arbitrary dimension, normal or Gorenstein:
  MCM $\implies$ Reflexive.

They are certainly not the same:

- For a one-dim Cohen Macaulay local ring with a canonical module ω, the ring is Gorenstein if and only if ω is reflexive.
- Any non Cohen Macaulay ring is a reflexive module over itself.

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- If they exist, are there "finitely" many of them in some sense? How to construct them?

Motivating questions

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Conjecture 1 [Hochster] Every complete local domain admits a maximal (small) Cohen Macaulay module.

- Clearly true if $dim(R) \leq 2$.
- There are examples of non catenary rings that do not admit a MCM module.
- (M.J. Bertin) Example of non CM UFDs that do not admit a MCM module of rank one.
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1. Conjecture 1 is known to be true only in very few cases.

Theorem 3 (Hartshorne,Hochster,Peskine-Szpiro)

Any three-dimensional $\mathbb{N}$-graded domain of characteristic $p > 0$ admits a maximal Cohen Macaulay module.

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Any three-dimensional pseudo-graded local ring of characteristic $p > 0$ admits a maximal Cohen Macaulay module.

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- Conjecture 1 reduces to the integral closure of a complete regular local ring in a finite Galois extension of its fraction field.

Theorem 5 (Roberts)

Let S be a regular local ring, L its quotient field, and K be a finite Abelian extension of L with Galois group G. Assume that the order of G is not divisible by the characteristic of the residue field of S. Then the integral closure of S in K is Cohen Macaulay.

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- The "modular case": that is when the degree of the extension is divisible by the characteristic of the residue field, the conclusion fails. (more on this later)
- Roberts's argument hinges on a nice decomposition of the group algebra $k[G]$ (Maschke's Theorem).
- On the other hand if K/L is a cyclic extension of degree p and $\mathrm{char}(k) = p$,

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- (Hochster, Roberts) Example of a local UFD S of mixed characteristic 2 such that its integral closure in a quadratic Abelian extension of its quotient field does not admit any non zero S-free module.
- Roberts showed his theorem is false if G is only assumed to be solvable or nilpotent.
- He shows there are Galois extensions of degree 8 of a characteristic zero regular local ring S, such that that the minimal number of generators of the integral closure R over S is arbitrarily large.

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The setting

- Extensions of the fraction field of an unramified regular local ring S with the property that the integral closure of S in such an extension is S-free if S contains a field but not necessarily so in the mixed characteristic scenario.
- Towards a mixed characteristic analog of Roberts's theorem: obstructions one faces when p divides the degree of the extension, with a two fold objective:
  1. Determining when the integral closure is Cohen Macaulay + bounds on the depth.
  2. When it is not Cohen Macaulay, determining whether it admits a maximal Cohen Macaulay module.

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- Radical extensions K/L obtained by adjoining n-th roots, where $p \mid n$.
- Kummer theory $\implies$ given an Abelian extension K/L with Galois group G, if L contains a n-th root of unity where n is the index of the Galois group, then $K = L(\sqrt[n]{a_1}, \ldots, \sqrt[n]{a_s})$ for some $a_i \in L$.
- (Huneke-Katz) When S has equicharacteristic zero or mixed characteristic q, $q \nmid n$, the integral closure of S in a radical tower of n-th roots is Cohen Macaulay, under some reasonable generality hypothesis.
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- (Koh) Example of an Abelian extension K/L of a (ramified) regular local ring S of mixed characteristic $p > 0$ with field of fractions L, such that the integral closure of S in K is not CM.
- (Katz) Example of a p-th root extension of an unramified regular local ring such that the integral closure is not CM.
- Koh and Katz examples: adjoining a p-th root of a non-square free element.
- (Katz) In case we adjoin a p-th root of a single square free element the integral closure is CM.
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Does the integral closure of a regular local ring of mixed characteristic $p > 0$ in a finite Abelian extension of its fraction field admit a maximal Cohen Macaulay module?

Theorem 7 (Katz)

Let S be an unramified regular local ring of mixed characteristic $p > 0$ with field of fractions L. Let $K := L(\omega)$ where ω is the p-th root of an arbitrary element of S. Let R be the integral closure of S in K. Then R admits a birational maximal Cohen Macaulay module.

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- Square free towers: "complexity" increases very fast.
- Intuition derived from a careful study of biradical extensions of an unramified regular local ring S of mixed characteristic $p > 0$ obtained by adjoining p-th roots of sufficiently general square free elements say $f, g \in S$.
- May think of it as the case where the Galois group is $\mathbb{Z}_p \times \mathbb{Z}_p$.

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Notation

- Let $(S, \mathfrak{m})$ denote an unramified regular local ring of mixed characteristic $p > 0$ and L be its field of fractions. Let $dim(S) = d \geq 3$.
- $f, g \in \mathfrak{m}$ square free, relatively prime elements or f, g units such that they are not p-th powers in S.
- $F(W) := W^p - f \in S[W]$ and $G(U) := U^p - g \in S[U]$, monic irreducible polynomials.
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- R integral closure of S in K, that is R is the integral closure of $A := S[\omega, \mu]$. Note A is CM and if $f, g \in \mathfrak{m}$:

$$A \simeq \frac{S[W, U]_{(\mathfrak{m}, W, U)}}{(F(W), G(U))}$$

- S^p is the subring of S obtained by lifting the image of the Frobenius map on S/p to S. Let $S^{p^k \wedge p^n}$ for $k, n \geq 1$ be the multiplicative subset of S of elements expressible in the form $x^{p^k} + yp^n$ for some $x, y \in S$.

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- Seen relatively easily that $A[1/p]$ is integrally closed.
- Thus in the ring A every height one prime not containing p is regular.
- There exists a unique height one prime $P \subseteq A$ containing p.

Proposition 8 (Sridhar)

With established notation:

- *R is CM if $f \notin S^p$ and $g \in S[\omega]^p$ (or vice versa).*
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Proposition 9 (Katz, Sridhar)

Let S be a complete unramified regular local ring of mixed characteristic p with perfect residue field. Then there exists a module finite extension of unramified regular local rings $(S, \mathfrak{m}) \subseteq (T, \mathfrak{n})$ such that $S \subseteq T^p$.

- Thus we may assume the elements whose roots we adjoin are in S^p. This allows us many advantages such as a better handle on R.
- Assume $f, g \in S^p$ and $f, g \notin pS$. Let $h_1, h_2 \in S$ represent p-th roots of f and g mod p respectively.

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$$
\begin{tikzcd}
S \arrow[r, hook, "finite"] \arrow[d, hook] & T \arrow[r, hook] \arrow[d, hook] & K' \arrow[dd, hook] \\
R \arrow[r, hook, "finite"'] \arrow[d, hook] & \mathscr{R} \arrow[dr, hook] & \\
K \arrow[rr, hook] & & \mathscr{K}
\end{tikzcd}
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Proposition 10 (Sridhar)

Let S be an unramified regular local ring of mixed characteristic p. The integral closure R of S in $K := L(\omega, \mu)$ is Cohen Macaulay if at least one of the rings $S[\omega], S[\mu]$ is not normal.

1. The cases when R is not CM occur when $S[\omega]$ and $S[\mu]$ are normal rings.
2. One might view this as unexpected, but roughly this happens due to the existence of an "unramified branch" over $S_{(p)}$, that is an unramified extension $S_{pS} \hookrightarrow R_Q$ for some height one prime $Q \subseteq R$.

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Theorem 11 (Sridhar)

Let S be an unramified regular local ring of mixed characteristic p. Assume that $S[\omega]$ and $S[\mu]$ are normal rings. If $fg^k \notin S^{p \wedge p^2}$ for $1 \leq k \leq p-1$ then R is CM.

Further, in this case $P^{(p-1)}$ is the conductor of R to A where P is the unique height one prime in A containing p and $P^{(p-1)}$ denotes the $(p-1)$-th symbolic power of P.

1. The condition that $fg^k \notin S^{p \wedge p^2}$ is saying that the ring

$$S[\omega\mu^k, \ldots, \omega^i \mu^{ki(mod)p}, \ldots, \omega^{p-1}\mu^{k(p-1)(mod)p}]$$

is normal. For example, if $p = 2$: $S[\omega\mu]$ is normal. If $p = 3$: $S[\omega\mu]$ and $S[\omega\mu^2, \omega^2\mu]$ are normal.

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- In mixed characteristic two, results are a little sharper since in this case such extensions are automatically Abelian.
- For odd primes p, primitive p-th root of unity in S ramifies p.

Proposition 12 (Sridhar)

Let $(S, \mathfrak{m})$ be an unramified regular local ring of mixed characteristic two. Let $S[\omega], S[\mu]$ be integrally closed rings and $fg \in S^{2\wedge 4}$.

1. *If $f, g \in \mathfrak{m}$, $R \simeq S^2 \bigoplus_S Syz_S^2(S/Q)$ where $Q := (2, h_1, h_2) \subset S$ and hence $p.d._S(R) \leq 1$.*
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Theorem 13 (Sridhar)

Let S be an unramified regular local ring of mixed characteristic 2 and dimension $d \geq 3$. Let $f, g \in S^2$.

1. *R is Cohen Macaulay if and only if one of the following happen*
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2. *If R is not Cohen Macaulay, R admits a birational maximal Cohen Macaulay module.*

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Corollary 14 (Katz, Sridhar)

Let S be a complete unramified regular local ring of mixed characteristic 2 and dimension $d \geq 3$. Assume that the residue field is perfect. Let $f, g \in S$ square free and relatively prime. Then R admits a maximal Cohen Macaulay module.

Theorem 15 (Sridhar)

Let $(S, \mathfrak{m})$ be an unramified regular local ring of mixed characteristic $p \geq 3$ and dimension $d \geq 3$. Then

1. *R is Cohen Macaulay if*
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2. *Let $S[\omega], S[\mu]$ be integrally closed such that $fg \in S^{p \wedge p^2}$ and $f, g \in \mathrm{m}$. Then R is Cohen Macaulay if and only if $Q := (p, h_1, h_2) \subset S$ is a grade two perfect ideal. Moreover, $p.d_S(R) \leq 1$ and $\nu_S(R) \leq p^2 + 1$. If f or g is a unit, then R is Cohen Macaulay.*
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- It appears R is not "too far" from being Cohen Macaulay, in the sense that $depth(R) \geq d - 1$ and it can be generated by $rank_S(R) + 1$ elements over the base ring S.
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- Studying the structure of the conductor J of the integral closure R to A.
- Since A is Gorenstein and J is unmixed, R is Cohen Macaulay if and only if A/J is Cohen Macaulay.
- To show that R admits a birational maximal Cohen Macaulay module we choose a suitable ideal $I \subseteq A$ such that I^* is a J^*-module and $depth_S(I^*) = d$.

Examples

- We can generate examples of non Cohen Macaulay R relatively easily.
- Let $S = \mathbb{Z}[X, Y]_{(p,X,Y)}$ for some prime number $p \geq 3$. Set

$$f = (X^2)^p + p(p - X^{2p})$$

$$g = (XY)^p + p(p + (XY)^p)$$

- Note that $f, g \in S^p$ with $h_1 = X^2$, $h_2 = XY$. Moreover f, g are square free and relatively prime in S.
- This gives rise to an example where $Q := (p, h_1, h_2)$ has grade two but $p.d_S(S/Q) = 3$ and hence R is not Cohen Macaulay.
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- When $f, g \notin S^p$, R is not necessarily CM.
- As an example, take $S := \mathbb{Z}[X, Y, V]_{(2,X,Y,V)}$ and $f = XV^2 + 4$, $g = XY^2 + 4$. Then f, g are square free, form a regular sequence in S and do not lie in S^2.
- Then it can be shown that R is CM if and only if

$$\frac{(\mathbb{Z}/2\mathbb{Z})[X, Y, V, W, U]_{(X,Y,V,W,U)}}{(W^2 - XV^2, U^2 - XY^2, UV - WY, WU - XYV)}$$

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- But they need not exist always:
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Let S be an unramified regular local ring of mixed characteristic $p > 0$ with fraction field L. Let $f_1, \ldots, f_n \in S^{p \wedge p^2}$, square free and mutually coprime. Let $\omega_i^{n_i} = f_i$ such that $p \mid n_i$ and $p^2 \nmid n_i$ for each i. If $f_i = p$, then assume $n_i = p$. Then the integral closure of S in $L(\omega_1, \ldots, \omega_n)$ is Cohen Macaulay.

- The above result enables the existence of small CM algebras for a broad class of non CM rings.
- One approach when S is complete with perfect residue field is to "reduce to S^p" and then ramify p suitably to expect behaviour similar to Theorem 16.

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Let S be an unramified regular local ring of mixed characteristic $p > 0$ with fraction field L. Let $f_1, \ldots, f_n \in S^{p \wedge p^2}$, square free and mutually coprime. Let $\omega_i^{n_i} = f_i$ such that $p \mid n_i$ and $p^2 \nmid n_i$ for each i. If $f_i = p$, then assume $n_i = p$. Then the integral closure of S in $L(\omega_1, \ldots, \omega_n)$ is Cohen Macaulay.

- The above result enables the existence of small CM algebras for a broad class of non CM rings.
- One approach when S is complete with perfect residue field is to "reduce to S^p" and then ramify p suitably to expect behaviour similar to Theorem 16.

Some Questions

S an unramified regular local ring of mixed characteristic $p > 0$ and dimension $d \geq 3$. Let L be its quotient field and K/L a finite field extension. Let R be the integral closure of S in K.

- If K/L is Abelian, does R admit a maximal Cohen Macaulay module/algebra?
- (Katz, Sridhar) A reduction to the p-torsion part of the Abelian group can be made under one exception.
- If R admits a (birational) maximal Cohen Macaulay module can it admit a small CM algebra?

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Reflexivity in codimension one

- Over a (S_2)-ring, reflexive $\iff$ (S_2) + reflexive in codimension one.
- Some general facts are known: for example if R is reduced, any second syzygy or R-dual module is reflexive.
- But reflexive modules or even ideals are not well understood even in the case of a one dimensional CM ring that is not Gorenstein.
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- R one dimensional analytically unramified Cohen Macaulay local ring. (typically not Gorenstein)
- We have
  1. $\{$free R- modules$\} \subset Ref(R) \subset CM(R)$.
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We say that $M \in CM(R)$ is I-Ulrich for a regular ideal I if $e_I(M) = \ell(M/IM)$. Let $Ul_I(R)$ denote the category of I-Ulrich modules.

- For a regular ideal I, let $B(I) := \cup_{n \geq 1} End(I^n)$ be the blow up ring of I. Let $M \in CM(R)$. The following are equivalent:
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- Nice algebraic properties such as if $M \in UI_I(R)$ then for any $N \in CM(R)$, $Hom_R(M, N) \in UI_I(R)$ and good behaviour along short exact sequences.
- I-Ulrich ideals form a lattice and the largest element is $b(I) := R : B(I)$.

Proposition 18 (Dao, Maitra, Sridhar)

Let R be a reduced one dimensional ring with infinite residue field k. Let I be a regular ideal with reduction number r. Assume that $char(k) = 0$ or $char(k) > r$. Then

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- Can we characterize such extensions?
- If this characterization is done in a "computable" manner, we can generate examples of reflexive modules.
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Suppose that R is a one-dimensional Cohen-Macaulay local ring with a canonical ideal ω_R. Let S be a finite extension of R. The following are equivalent (for the last implication, assume that S is a birational extension and the residue field of R is infinite):

1. *Any maximal Cohen-Macaulay S-module is R-reflexive.*
2. *ω_S is R-reflexive.*
3. *$Hom_R(S, R) \cong Hom_R(S, \omega_R)$.*
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Corollary 21

Assume R has a canonical ideal ω_R. Let $Q(R) \hookrightarrow A$ be an extension of the total quotient ring of R. If the integral closure of R in A, say $\bar{R}^A$, is a finite R-module, then $\bar{R}^A \in Ref(R)$.

Corollary 22

Let S be a module finite R-algebra such that R is a generically Gorenstein (S_2) ring of arbitrary dimension and S is (S_1). If $R \to S$ is strongly reflexive in codimension one, then any finite (S_2) S-module M is R-reflexive.

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A class of objects is said to have **finite type** if there are only finitely many indecomposable objects up to isomorphism.

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Let R be a one dimensional analytically unramified Cohen Macaulay local ring and $\mathfrak{c}$ its conductor ideal. Further assume that k is infinite or $|Min(\hat{R})| \leq |k|$. Consider the following.

1. $\ell(R/\mathfrak{c}) \leq 3$
2. $\ell(R/\mathfrak{c}) = 4$ *and R has minimal multiplicity.*

Then in all the above cases, $Ref_1(R)$ is of finite type.

- The above theorem is sharp: $R = k[[t^4, t^5, t^6]]$ is a complete intersection domain of multiplicity 4 (not minimal multiplicity), $\ell(R/\mathfrak{c}) = 4$ but $Ref_1(R)$ is not of finite type.

Theorem 26 (Dao, Maitra, Sridhar)

Assume that $(R, \mathfrak{m})$ is a one dimensional, reduced complete local ring and the conductor $\mathfrak{c}$ of R is equal to $\mathfrak{m}$. Then $Ref(R)$ is of finite type.

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- More results in the case of **Almost Gorenstein rings** ($\mathfrak{m}$ is ω_R-Ulrich). For example in such rings all powers of trace ideals are reflexive.

Definition 27

The *trace ideal* of an R-module M (denoted $tr(M)$) is the image of the natural map $\tau_M : M^* \otimes_R M \to R$. An ideal $I \subseteq R$ is a trace ideal if $I = tr(I)$.

- Examples: $\mathfrak{m}$, $\mathfrak{c}$, $\mathfrak{c}_R(S)$ for any finite birational extension S.
- In general trace ideals need not be reflexive.
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Let R be a complete local or graded ring. Are the following equivalent?

1. *$CM(R)$ is of finite type.*
2. *There are only finitely many possibilities for $tr(M)$, where $M \in CM(R)$.*

- Let $R = k[[t^e, \dots, t^{2e-1}]]$ where k is infinite and $e \geq 4$. Then the set of trace ideals is finite but $CM(R)$ is infinite.

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Let R be a complete Cohen Macaulay local ring of dimension one. Can we classify when R has finitely many trace ideals?

- We do this for up to three trace ideals when R contains a field.

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Thank you !