How valuation rings behave like regular rings

Based on joint works with Karen Smith and Benjamin Antieau.

Throughout the talk K is a field.

A valuation ring V of K is a subring such that

$$\forall x \in K, \quad x \in V \text{ or } x^{-1} \in V.$$

$\boxed{\text{Some properties}}$ A valuation ring V is

① local with max'l ideal $\mathfrak{m}_V = \{x \in K : x^{-1} \notin V\}$.

② normal

③ $\forall x, y \in V$, $x \mid y$ or $y \mid x$.

[If $x, y \neq 0$, then $yx^{-1} \in V$ or $xy^{-1} \in V$]

④ finitely gen. ideals are principal.

⑤ ideals are totally ordered by $\subseteq$.

$\therefore$ Spec V is a chain of primes.

First sign that valuation rings are like regular rings.

Lemma A valuation ring V of K is noetherian $\Rightarrow$ $V = K$ or V is regular of $\dim = 1$.

Pf: Assume $V \neq K$.

Every f.g. ideal of V is principal $\Rightarrow$ V is a PID.

A local PID is regular of $\dim = 1$

AKA

a DVR.

$\square$

Upshot Valuation rings are usually non-noetherian.

Examples

① A is noetherian + normal $\Rightarrow$ $\forall\, \mathfrak{p} \in \operatorname{Spec} A$ s.t. $\operatorname{ht} \mathfrak{p} = 1$, $A_{\mathfrak{p}}$ is a val. ring.

gives rise to theory of divisors in AG.

② the p-adic integers $\mathbb{Z}_p$.

non-noetherian valuation rings arise naturally.

③ Let $\mathbb{Q}_p = \mathrm{Frac}(\mathbb{Z}_p)$.

The integral closure $\overline{\mathbb{Z}}_p$ of $\mathbb{Z}_p$ in $\overline{\mathbb{Q}}_p$ is a non-noeth. val. ring of dim 1.

$\widehat{\overline{\mathbb{Z}}_p}^{(p)}$ = a val. ring;
integral perfectoid algebra.

This is a fundamental object in p-adic Hodge theory.

MAIN THEOREM [Antieau - D]:

Valuation rings are derived splinters.

Compare this with

A. Direct Summand Theorem [Hochster, André]:

Regular rings are splinters.

B. Derived direct summand theorem [Bhatt]:

Regular rings are derived splinters.

Further evidence that valuation rings behave like

REGULAR rings.

Splinters A ring R is a splinter if ANY finite $\varphi : R \to S$ surjective on Spec is pure i.e.

$\forall$ R-mods M, $\varphi \otimes id_M : M \to S \otimes_R M$ is injective.

Example If $\varphi : R \to S$ splits, it's pure.

Lemma Let V be a valuation ring. Any fin. gen. torsion free V-mod is free.

Pf: Let M be finitely generated + torsion free.

Assume $M \neq 0$. Choose a minimal gen set

$$\{m_1, \dots, m_n\}, \quad \text{where } n \geq 1.$$

Claim: $\{m_1, \dots, m_n\}$ is free.

If not, $\exists$ $x_1, \dots, x_n \in V$ not all 0 s.t.

$$x_1 m_1 + \dots + x_n m_n = 0.$$

V is a valuation ring $\Rightarrow$ wlog $x_1 \mid x_i$ for ALL i.

$$\Rightarrow m_1 = \frac{-x_2}{x_1} m_2 - \dots - \frac{x_n}{x_1} m_n.$$

Contradicts minimality.

$\square$

Compare with

First result in structure theory of modules over a PID:

A f.g. torsion free module over a PID is free

$\boxed{\text{Corollary}}$ A torsion free module over a valuation ring is flat.

Pf: Express module as a filtered union of f.g. submodules, which are free hence flat.

$\square$

Exercise: V is a valuation ring + M is a finitely presented V-mod $\Rightarrow$ $\text{proj. dim}_V M \leq 1$.

$\boxed{\text{Theorem [D]}}$ Valuation rings are splinters.

Pf: Let V be a valuation ring. Suppose

$$\varphi : V \to S$$

is finite + surjective on Spec.

Choose $p \in \operatorname{Spec} S$ s.t. $\varphi^{-1}(p) = (0)$.

Composition $V \to S \twoheadrightarrow S/p$ is finite + injective.

S/p is a domain $\Rightarrow$ S/p is V torsion-free.

$\therefore$ S/p is free, hence $V \xrightarrow{\varphi} S \twoheadrightarrow S/p$ splits. So does φ.

$\square$

Brief digression: Suppose A has prime char. $p > 0$.

Recall

$\boxed{\text{Kunz's Thm}}$ If A is noeth, A is regular
$\iff$ $F : A \to A$ is flat.
$x \mapsto x^p$

$\boxed{\text{Valuative Kunz's Thm [D-Smith]}}$ For a valuation ring V of char $p > 0$,

$$F : V \to V$$

is flat.

Pf: Target copy of V is torsion free as a module over the domain, hence flat.

$\square$

As a consequence also obtain valuation rings in prime char. are F-pure.

Smith and I used this observation to build a theory of F-singularities of valuations.

$\boxed{\text{Derived splinters}}$ For a ring A,

$D(A)$ = derived cat. of complexes of A-mods.

Morphisms are complicated ... Chain maps that induce isos on cohomology are invertible in $D(A)$.

Let $X \xrightarrow{f} \operatorname{Spec} A$ be a morphism of schemes.

$\boxed{R\Gamma(X, \mathcal{O}_X)}$ Take an injective resolution in $\mathrm{Mod}_{\mathcal{O}_X}$

$$0 \to \mathcal{O}_X \longrightarrow \mathcal{I}^0 \to \mathcal{I}^1 \to \cdots$$

$$R\Gamma(X, \mathcal{O}_X) := 0 \to \Gamma(X, \mathcal{I}^0) \to \Gamma(X, \mathcal{I}^1) \to \cdots$$
$$= \text{complex of } A\text{-mods.}$$

$$H^0(R\Gamma(X, \mathcal{O}_X)) = \ker(\Gamma(X, \mathcal{I}^0) \to \Gamma(X, \mathcal{I}^1)) = \Gamma(X, \mathcal{O}_X)$$

$\boxed{\text{Example}}$ f is finite $\Rightarrow R\Gamma(X, \mathcal{O}_X) \simeq \Gamma(X, \mathcal{O}_X)$.

Let

$$f^\# := A \longrightarrow \Gamma(X, \mathcal{O}_X) \longrightarrow R\Gamma(X, \mathcal{O}_X).$$

A is a derived splinter if $\forall$ proper, surjective, finitely presented [as algebras]

$$f : X \longrightarrow \operatorname{Spec} A$$

$f^\#$ has a left-inverse in $D(A)$.

$\boxed{\text{Examples}}$ ① A is finite type $/\mathbb{C}$, A is a D-splinter

$\iff$ A has rational sing. [Kovács]

② A is noeth. + char $p > 0$, A is a D-splinter $\iff$ A is a splinter [Bhatt]

$\boxed{\text{WANT}}$ A valuation ring V is a D-splinter.

Today: Sketch proof when V is absolutely integrally closed (a.i.c) i.e.

$$\mathrm{Frac}(V) = \text{algebraically closed}.$$

$\boxed{\text{Idea}}$ V is a.i.c $\rightsquigarrow$ regular rings approximate V

$\rightsquigarrow$ reduce to Bhatt's derived direct summand

de Jong's theorem on alterations $\Rightarrow$

if V is an a.i.c. valuation ring over $k = \mathbb{Q}, \mathbb{F}_p$ or $\mathbb{Z}$

then

$$V = \text{filtered colimit of finite type regular } k\text{-subalgebras}$$

$\boxed{\text{Upshot}}$ Write

$$V = \operatorname{colim}_i A_i, \quad A_i = \text{regular}.$$

$f : X \to \operatorname{Spec} V$ proper + surjective + finitely presented

$\Rightarrow \exists\ i$ and $f_i : X_i \to \operatorname{Spec} A_i$

proper + surjective s.t. $X \cong X_i \times_{\operatorname{Spec} A_i} \operatorname{Spec} V$

That is we have a Cartesian square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X_i \arrow[d] \\ \mathrm{Spec}\, V \arrow[r] & \mathrm{Spec}\, A_i \end{tikzcd}.$$

Derived direct summand $\Rightarrow$ $A_i \longrightarrow R\Gamma(X_i, \mathcal{O}_{X_i})$

splits in $D(A_i)$

$$\Rightarrow \quad \underset{\substack{\parallel \\ V}}{A_i \otimes^L_{A_i} V} \longrightarrow R\Gamma(X_i, \mathcal{O}_{X_i}) \otimes^L_{A_i} V$$

splits in $D(V)$.

Would win if $R\Gamma(X_i, \mathcal{O}_{X_i}) \otimes^L_{A_i} V \simeq R\Gamma(X, \mathcal{O}_X)$.

But life is unfair :(

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & X_i \arrow[d, "f_i"] \\ \mathrm{Spec}\, V \arrow[r] & \mathrm{Spec}\, A_i \end{tikzcd}$$

may not be Tor-independent.

One way to ensure $R\Gamma(X_i, \mathcal{O}_{X_i}) \otimes^L_{A_i} V \simeq R\Gamma(X, \mathcal{O}_X)$: Make f_i flat.

If f is flat, can choose i s.t. f_i is flat.
(in addition to being proper + surjective)

$\boxed{\text{Making } f \text{ flat}}$ $\qquad \underbrace{J := \sum_{r \in V - \{0\}} \ker\left(\mathcal{O}_X \xrightarrow{r\cdot} \mathcal{O}_X\right)}$

V-torsion ideal sheaf of X.

$\therefore \quad \mathbb{V}(J) \subseteq X \xrightarrow{f} \operatorname{Spec} V$

is

- flat (killed torsion)
- proper
- surjective (not hard)
- $\underbrace{\text{finitely presented}}$

Raynaud - Gruson miracle flatness :

$A \to B$ finite type + flat + A is domain $\Rightarrow$ $A \to B$ is of fin. presentation.

$\boxed{\text{Upshot}}$ $\quad \mathbb{V}(J) \subseteq X \xrightarrow{f} \operatorname{Spec} V$ gives a composition

$$V \to R\Gamma(X, \mathcal{O}_X) \to R\Gamma(\mathbb{V}(J), \mathcal{O}_{\mathbb{V}(J)})$$

which splits. Hence so does $V \to R\Gamma(X, \mathcal{O}_X)$.

$\square$