Stable Harbourne–Huneke Containment and Bounds on Waldchmidt Constant

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Motivation: Polynomial Interpolation Problem

- Polynomials in one variable are completely determined by its zeros.
- General: Given s distinct points $\mathscr{X} = \{x_1, \ldots, x_s\}$ in the affine line $\mathbb{A}^1_{\mathbb{C}}$, a polynomial of degree d, where $d + 1 = \sum m_i$

$$f(x) = a_0 + a_1 x + \cdots + a_d x^d$$

vanishing on $\mathscr{X}$ with multiplicity $m_1, \ldots, m_s$ respectively, can be uniquely determined with the following vanishing condition of the derivatives

$$f^{(j)}(x_i) = 0 \text{ for } i = 1, \ldots, s \text{ and } j = 0, \ldots, m_i - 1.$$

- You may know this by Hermit Interpolation

Motivation: Polynomial Interpolation Problem

- What happens with polynomial with several variable or in higher dimensions?
- $\mathbb{K} = \mathbb{C}$, $\mathbb{P}^N_{\mathbb{C}} = \mathbb{P}^N$, $m_1, \dots, m_s \in \mathbb{N}$, $\mathbb{X} = \{P_1, \dots, P_s\} \subset \mathbb{P}^N$

Polynomial Interpolation Problem

Given $\mathbb{X} = \{P_1, \dots, P_s\} \subset \mathbb{P}^N$ and positive integers $m_1, \dots, m_s$.

$$\mathscr{L}_{\mathbb{X},m_i} = \Big\{ \text{homogeneous polynomials vanishing on } \mathbb{X} \\ \text{with multiplicities } m_1, \dots, m_s \Big\} \subset \mathbb{C}[x_0, \dots, x_N]$$

1. $\mathscr{L}_{\mathbb{X},m_i}$ is an algebraic object connecting geometric properties of $\mathbb{X}$.
2. What is the minimal $d \in \mathbb{N}$ such that $\mathscr{L}_{\mathbb{X},m_i}$ has a polynomial of degree d?
3. Fix any $d \in \mathbb{N}$. What is $\dim_{\mathbb{C}}\{f \in \mathscr{L}_{\mathbb{X},m_i} \mid \deg(f) = d\}$?

Our Focus: Equi-multiplicity & Nagata's Question

- Equi-multiplicity: $m_1 = \cdots = m_s = m$
- What is the minimal $d \in \mathbb{N}$ such that $\mathscr{L}_{\mathbb{X},m}$ has a polynomial of degree d?
- Small Note: Hyper-surfaces are defined by homogeneous polynomials.

Question 1.1 (Nagata'65)

Take a set of points $\mathbb{X} = \{P_1, \ldots, P_s\} \subset \mathbb{P}^2_{\mathbb{C}}$. What is the minimal degree $\alpha_t(\mathbb{X})$ of a hypersurface that passes through the points with multiplicity at least t?

Goal: To study lower bounds of $\alpha_t(\mathbb{X})$.

Conjectural Bounds

Conjecture 1.1 (Nagata)

Let $\mathbb{X}$ be a general set of s points in $\mathbb{P}^2$, then

$$\frac{\alpha_t(\mathbb{X})}{t} \geqslant \sqrt{s}$$

Conjecture 1.2 (Chudnovsky and Demailly)

If $\mathbb{X} = \{P_1, \ldots, P_s\} \subset \mathbb{P}^N_{\mathbb{C}}$, and m, t be positive integers then

- *(Chudnovsky)*

$$\frac{\alpha_t(\mathbb{X})}{t} \geq \frac{\alpha(\mathbb{X}) + N - 1}{N}, \ \forall t \geq 1, \ \alpha(\mathbb{X}) = \alpha_1(\mathbb{X}).$$

- *(Demailly)*

$$\frac{\alpha_t(X)}{t} \geq \frac{\alpha_m(X) + N - 1}{m + N - 1}, \ \forall m, \ t \geq 1.$$

Zariski-Nagata Theorem

Definition 1.1 (Powers of ideals)

Let I is an ideal in $\mathbb{K}[x_0, \ldots, x_N]$,

- Symbolic Powers: $I^{(t)} = \bigcap_{\mathfrak{p} \in \mathrm{Ass}(I)} (I^t S_{\mathfrak{p}} \cap S)$.
- Differential: $I^{\langle t \rangle} = \{f \in S : \partial_{\underline{\alpha}}(f) \in I \text{ for } |\underline{\alpha}| < t\}, \partial \equiv$ "partial".

Theorem 1.2 (Zariski-Nagata 1949-62)

Let $S = \mathbb{K}[x_0, \ldots, x_N]$ where $\mathbb{K}$ be a perfect field and let, I is a radical ideal then $I^{(t)} = I^{\langle t \rangle}$ for all $t \geq 1$.

Fat Points in $\mathbb{P}^N$

$\mathbb{X} = \{P_1, \ldots P_s\}$, $\mathbb{Y} = m_1 P_1 + \cdots + m_s P_s$(fat points), $\mathfrak{p}_i$ ideal defining P_i.
Then, $I_{\mathbb{X}}^{(m)} = \mathfrak{p}_1^m \cap \cdots \cap \mathfrak{p}_s^m = \mathscr{L}_{\mathbb{X},m}$ and $I_{\mathbb{Y}} = \mathfrak{p}_1^{m_1} \cap \cdots \cap \mathfrak{p}_s^{m_s} = \mathscr{L}_{\mathbb{X},m_i}$

Waldschmidt Constant

So, $\alpha_t(\mathbb{X}) = \alpha(I_{\mathbb{X}}^{(t)})$ [by Zariski-Nagata],
where, $\alpha(I_{\mathbb{X}}^{(t)}) \equiv$ the initial degree of $I_{\mathbb{X}}^{(t)}$, $I_{\mathbb{X}}$ is the ideal defining $\mathbb{X}$.

Subadditivity of $\alpha(I^{(t)})$

$I^{(a)}I^{(b)} \subset I^{(a+b)}$ implying $\alpha(I^{(a)}) + \alpha(I^{(b)}) \geq \alpha(I^{(a+b)}) \implies \{\alpha(I^{(t)})\}$ is sub-additive for any ideal I.

Definition 1.3 (Waldschmidt Constant)

Hence, by Fekete's lemma,

$$\widehat{\alpha}(I) = \lim_{t\to\infty} \frac{\alpha(I^{(t)})}{t} = \inf \frac{\alpha(I^{(t)})}{t}$$

is well defined.

Equivalent form

Conjecture 1.3 (Nagata's)

If $\mathbb{X}$ be any set of s general points in $\mathbb{P}^2$, and $I_{\mathbb{X}}$ be the defining ideal then for all $t \geq 1$,

$$\frac{\alpha(I_{\mathbb{X}}^{(t)})}{t} \geq \sqrt{s}. \text{ That is, } \widehat{\alpha}(I_{\mathbb{X}}) \geq \sqrt{s}$$

Conjecture 1.4 (Chudnovsky's Equivalent form)

Let $I_{\mathbb{X}}$ be the defining ideal of a finite set of points $\mathbb{X} \subset \mathbb{P}^N$, then,

- *(Chudnovsky)*

$$\widehat{\alpha}(I_{\mathbb{X}}) \geq \frac{\alpha(I_{\mathbb{X}}) + N - 1}{N}.$$

- *(Demailly)*

$$\widehat{\alpha}(I_{\mathbb{X}}) \geq \frac{\alpha(I_{\mathbb{X}}^{(m)}) + N - 1}{m + N - 1}. \ \forall m \geq 1$$

Results: Chudnovsky's Conjecture

Previous and New

- Any finite set of points in $\mathbb{P}^2_{\mathbb{C}}$ by Chudnovsky'81 also by Harbourne and Huneke'13.
- Any finite set of general points in $\mathbb{P}^3_{\mathbb{K}}$ and a finite set of at most $N+1$ general points in $\mathbb{P}^N_{\mathbb{K}}$ by Dumnicki 2012.
- Any set of binomial number of points in $\mathbb{P}^N_{\mathbb{K}}$ forming a star configuration by Bocci and Harbourne 2010.
- Any set of more than 2^N very general points in $\mathbb{P}^N_{\mathbb{K}}$ by Dumnicki and Tutaj-Gasińska 2016.
- Any finite set of very general points in $\mathbb{P}^N_{\mathbb{K}}$ by Fouli, Mantero and Xie 2016.
- At least 4^N many general points in $\mathbb{P}^N_{\mathbb{C}}$ (–, Grifo, Hà, Nguyễn 2020)

Results: Demailly's Conjecture

Previous and New

- Esnault and Viehweg, 1983, proved, for points in $\mathbb{P}^N_{\mathbb{C}}$.
- Malara, Szemberg and Szpond 2018 proved for a fixed integer m, Demailly's Conjecture holds for s number of very general points in $\mathbb{P}^N_{\mathbb{K}}$, where $\lfloor \sqrt[N]{s} \rfloor \geq m+1$.
- Extended by Chang and Jow 2020, showed that for a fixed integer m, Demailly's Conjecture holds for s number of very general point $\mathbb{P}^N_{\mathbb{K}}$, where $\lfloor \sqrt[N]{s} \rfloor - 2 \geq \dfrac{2\varepsilon}{N-1}(m-1)$, $0 \leq \varepsilon < 1$.
- At least $(2m+2)^N$ many general points in $\mathbb{P}^N_{\mathbb{C}}$ (–, Grifo, Hà, Nguyễn 2020)

Points in $\mathbb{P}^N$

- $\mathbb{X} = \{P_1, \ldots, P_s\}$, where

$$
\begin{aligned}
P_1 &= [a_{10} : a_{11} : \cdots : a_{1N}] \\
P_2 &= [a_{20} : a_{21} : \cdots : a_{2N}] \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
P_s &= [a_{s0} : a_{s1} : \cdots : a_{sN}]
\end{aligned}
$$

- $\mathbb{X}$ is associated the matrix

$$
\begin{bmatrix} a_{10} & a_{11} & \ldots & a_{1N} \\ a_{20} & a_{21} & \ldots & a_{2N} \\ \ldots & \ldots & \ldots & \ldots \\ a_{s0} & a_{s1} & \ldots & a_{sN} \end{bmatrix} \in \mathbb{A}^{s(N+1)}, \text{ and } \begin{bmatrix} z_{10} & z_{11} & \ldots & z_{1N} \\ z_{20} & z_{21} & \ldots & z_{2N} \\ \ldots & \ldots & \ldots & \ldots \\ z_{s0} & z_{s1} & \ldots & z_{sN} \end{bmatrix}
$$

Generic Points

- Let $\mathrm{z} = (z_{ij})_{1 \leqslant i \leqslant s, 0 \leqslant j \leqslant N}$ be $s(N+1)$ new indeterminates.
- Let $\mathbb{K}(\mathrm{z})$ be the transcendental extension of $\mathbb{K}$.
- For $i = 1, \ldots, s$, set $P_i(\mathrm{z}) = [z_{i0} : \cdots : z_{iN}] \in \mathbb{P}^N_{\mathbb{K}(\mathrm{z})}$

Definition 1.4

The set $\mathbb{X}(\mathrm{z}) = \{P_1(\mathrm{z}), \ldots, P_s(\mathrm{z})\}$ is referred to as the set of s *generic points* in $\mathbb{P}^N_{\mathbb{K}(\mathrm{z})}$.

Remark 1.1

Let $\mathrm{a} = (a_{ij})_{1 \leqslant i \leqslant s, 0 \leqslant j \leqslant N} \in \mathbb{A}^{s(N+1)}$. By replacing z with a $(z_{ij} \leftrightarrow a_{ij})$, we can find an open dense subset $W \subset \mathbb{A}^{s(N+1)}$ such that $\mathbb{X}(\mathrm{a}) = \{P_1(\mathrm{a}), \ldots, P_s(\mathrm{a})\}$ is set of distinct s points in $\mathbb{P}^N$ for $\mathrm{a} \in W$.

General and Very General Points in $\mathbb{P}^N_{\mathbb{K}}$

By Fouli-Mantero-Xie:

A property $\mathscr{P}$ is said to hold for	
Very General set of s Points if	General set of s Points if
there exists infinite intersection of open sets $W = \cap_{i=1}^{\infty} W_i \subseteq \mathbb{A}_{\mathbb{K}}^{s(N+1)}$ such that the property $\mathscr{P}$ holds for $\mathbb{X}(a)$ whenever $a \in W$.	there exists open $W \subseteq \mathbb{A}_{\mathbb{K}}^{s(N+1)}$ such that the property $\mathscr{P}$ holds for $\mathbb{X}(a)$ whenever $a \in W$.

Containment Problem

Theorem 2.1 (Ein-Lazarsfeld-Smith'01, Hochster-Huneke'02 , and Ma-Schwede'17)

For a radical ideal I of bigheight h in a regular ring S, one has

$$I^{(hr)} \subseteq I^r$$

for all $r \in \mathbb{N}$, which implies: $\widehat{\alpha}(I) \geq \frac{\alpha(I)}{h}$

Conjecture 2.1 (Harbourne'09)

For a radical ideal I of bigheight h in a regular ring S,

$$I^{(hr-h+1)} \subseteq I^r \ \forall r \geq 1.$$

Harbourne-Huneke Conjectures

$$I^{(hr-h+1)} \subseteq I^r \ \forall r \geq 1.$$

Conjecture 2.2 ([Harbourne-Huneke'13)

] Let $R = \mathbb{K}[\mathbb{P}^N_{\mathbb{K}}]$, $I \subset R$, *homogeneous radical with bigheight* $= h$, *and* $\mathfrak{m} = \langle x_0, \ldots, x_N \rangle$. *Then for* $r \geq 1$

1. $I^{(hr)} \subseteq \mathfrak{m}^{r(h-1)} I^r$, *and*
2. $I^{(r(m+h-1))} \subseteq \mathfrak{m}^{r(h-1)} (I^{(m)})^r$.

Points – Our focus

To study these containment for ideals defining points in $\mathbb{P}^N_{\mathbb{K}}$.

If $I \subset \mathbb{K}[\mathbb{P}^N_{\mathbb{K}}]$ is an ideal defining points, then the bigheight $h = N$.

Containment implies Chudnovsky and Demailly

Lemma

The Harbourne-Huneke containment implies Chudnovsky's conjecture and Demailly's conjecture.

Chudnovsky's Conjecture

Let $I \subset \mathbb{K}[\mathbb{P}^N_{\mathbb{K}}]$ defines an ideal defining a set of points in $\mathbb{P}^N_{\mathbb{K}}$

$$\text{Now, } I^{(Nr)} \subseteq \mathfrak{m}^{r(N-1)} I^r \implies \alpha(I^{(Nr)}) \geq r(\alpha(I) + N - 1)$$
$$\implies \frac{\alpha(I^{(Nr)})}{Nr} \geq \frac{\alpha(I) + N - 1}{N}$$

Taking limit as $r \to \infty$, we get $\widehat{\alpha}(I) \geq \frac{\alpha(I)+N-1}{N}$

Similarly, $I^{(r(m+h-1))} \subseteq \mathfrak{m}^{r(h-1)}(I^{(m)})^r$ implies Demailly's conjecture.

Remark 2.1

1. *Counter examples to Harbourne's conjecture exits (Dumniscki, Sxemberg and Tutaj-Gasinska'13; Harbourne Seceleanu'15).*
2. *Due to the limiting scenario, we can take $\forall r \gg 0$ and concentrate on stable containment.*

Conjecture 2.3 (Stable Containment)

Let $R = \mathbb{K}[\mathbb{P}^N_{\mathbb{K}}]$, $I \subset R$, homogeneous radical with bigheight $= h$, and $\mathfrak{m} = \langle x_0, \dots, x_N \rangle$. Then

1. $I^{(hr-h+1)} \subseteq I^r$, *for $r \gg 0$ [Stable Harbourne]*
2. $I^{(hr)} \subseteq \mathfrak{m}^{r(h-1)} I^r$, *for $r \gg 0$ [Stable Harbourne-Huneke]*
3. $I^{(r(m+h-1))} \subseteq \mathfrak{m}^{r(h-1)}(I^{(m)})^r$, *where $r = ct$ for some fix $c \in \mathbb{N}$ and for all $t \in \mathbb{N}$. (weaker stable HH)*

Results: Containment and Stable Containment

- Tohaneanu, and Xie'19 proved stable Harbourne containment $I^{(Nr-N+1)} \subset I^r$ for very general set of points in $\mathbb{P}^N$.
- Harbourne, and Huneke'13 proved $I^{(Nr)} \subset I^r \mathfrak{m}^{r(N-1)}$ for ideals defining any number of general points in plane, meaning $N = 2$.
- Dumnicki and Tutaj-Gasińka'16 proved Harbourne-Huneke containment $I^{(Nr)} \subset I^r \mathfrak{m}^{r(N-1)}$ for ideals defining $s \geq 2^N, N \geq 3$ many very general points.
- Bocci, Cooper, and Harbourne'14 proved $I^{(r(m+N-1))} \subseteq \mathfrak{m}^{r(N-1)}(I^{(m)})^r$ for ideals defining s^2 many general points in projective plane.

Results

Theorem 2.2 (–, Grifo, Hà, Nguyễn)

1. *The stable Harbourne conjecture holds for the ideal defining a general set of points in $\mathbb{P}^N$.*
2. *The stable Harbourne-Huneke conjecture*

$$I^{(Nr)} \subseteq \mathfrak{m}^{r(N-1)} I^r$$

holds for the ideal defining at least 4^N general set of points in $\mathbb{P}^N$.

3. *The Harbourne-Huneke containment*

$$I^{(r(m+N-1))} \subseteq \mathfrak{m}^{r(N-1)} (I^{(m)})^r$$

holds for $r = ct$, where $c \in \mathbb{N}$ is fixed and $t \in \mathbb{N}$, for the ideal defining at least $(2m+2)^N$ many general points in $\mathbb{P}^N$.

Specialization and Example

Definition 2.3 (Krull)

Let x represent the coordinates $x_0, \ldots, x_N$ of $\mathbb{P}^N_{\mathbb{K}}$, and $\mathrm{z} = (z_{ij})_{1 \leqslant i \leqslant s, 0 \leqslant j \leqslant N}$. Let $\mathrm{a} \in \mathbb{A}^{s(N+1)}$. The *specialization* at a is a map π_{a} from the set of ideals in $\mathbb{K}(\mathrm{z})[\mathrm{x}]$ to the set of ideals in $\mathbb{K}[\mathrm{x}]$, defined by

$$\pi_{\mathrm{a}}(I) := \{ f(\mathrm{a}, \mathrm{x}) \mid f(\mathrm{z}, \mathrm{x}) \in I \cap \mathbb{K}[\mathrm{z}, \mathrm{x}] \}.$$

Example 2.4

Consider the ideal $I = (x, y) = (x, x + uy)$ in $\mathbb{K}(u)[x, y]$. Then the specialization $\pi_0(I) = (x, y) \neq (x, x + 0y) = (x)$ in $\mathbb{K}[x, y]$.

Specialization with Ideal operations

Theorem 2.5 (Krull)

Let I, J be ideals of $\mathbb{K}(\mathrm{z})[\mathrm{x}]$.

1. *If $I = (f_1(\mathrm{z},\mathrm{x}), \dots, f_l(\mathrm{z},\mathrm{x}))$. Then there exists an open dense subset $W \subset \mathbb{A}_{\mathbb{K}}^{s(N+1)}$ such that for all $\mathrm{a} \in W$, $\pi_{\mathrm{a}}(I) = (f_1(\mathrm{a},\mathrm{x}), \dots, f_l(\mathrm{a},\mathrm{x}))$.*
2. *There exists an open dense subset $U \subset \mathbb{A}^{s(N+1)}$ such that for all $\mathrm{a} \in U$,*

$$\pi_{\mathrm{a}}(I \cap J) = \pi_{\mathrm{a}}(I) \cap \pi_{\mathrm{a}}(J) \text{ and } \pi_{\mathrm{a}}(IJ) = \pi_{\mathrm{a}}(I)\pi_{\mathrm{a}}(J)$$

Example 2.6

1. Let $I = (x_1)$, and $J = (x_1 + zx_2)$ be ideal in $\mathbb{K}(z)[x_1, x_2]$. Then the computations shows that, $\pi_0(I \cap J) = (x_1^2) \subsetneq \pi_0(I) \cap \pi_0(J) = (x_1)$
2. Let $I = (x_1, x_2 + zx_3)$ and $J = (x_1, x_2)$ be ideals in $\mathbb{K}(z)[x_1, x_2, x_3]$; $IJ = (x_1^2, x_1x_2, x_1x_3, x_2^2 + zx_2x_3)$; Now, $\pi_0(I).\pi_0(J) = (x_1, x_2)^2 \subsetneq \pi_0(I.J) = (x_1^2, x_1x_2, x_2^2, x_1x_3)$.

Open set in $\mathbb{A}_{\mathbb{K}}^{s(N+1)}$

Observation

- Let $\mathfrak{p}_i(\mathrm{z})$ and $\mathfrak{p}_i(\mathrm{a})$ be the defining ideals of $P_i(\mathrm{z}) \in \mathbb{P}^N_{\mathbb{K}(\mathrm{z})}$ and $P_i(\mathrm{a}) \in \mathbb{P}^N_{\mathbb{K}}$, respectively. Then there exists an open dense subset $W \subseteq W_0 \subseteq \mathbb{A}^{s(N+1)}$ such that, for all $\mathrm{a} \in W$ and any $1 \leqslant i \leqslant s$, we have
$$\pi_{\mathrm{a}}(\mathfrak{p}_i(\mathrm{z})) = \mathfrak{p}_i(\mathrm{a}).$$
- Let $I(\mathrm{z})$ and $I(\mathrm{a})$ be defining ideal of $\mathbb{X}(z)$ and $\mathbb{X}(\mathrm{a})$ respectively. For fixed $m, r, t \in \mathbb{N}$, there exists an open dense subset $U_{m,r,t} \subseteq W$ such that for all $\mathrm{a} \in U_{m,r,t}$, we have
$$\pi_{\mathrm{a}}\left(I(\mathrm{z})^{(m)}\right) = I(\mathrm{a})^{(m)} \text{ and } \pi_{\mathrm{a}}\left(\mathfrak{m}_{\mathrm{z}}^t I(\mathrm{z})^r\right) = \mathfrak{m}^t I(\mathrm{a})^r.$$

From Very general to General: The missing Bridge

Theorem 2.7 (Fouli, Mantero, and Xie'18)

$$\frac{\alpha(I^{(m)}_{\mathbb{X}(z)})}{m} \geq \frac{\alpha(I_{\mathbb{X}(z)} + N - 1)}{N} \implies \frac{\alpha(I^{(m)}_{\mathbb{X}(a)})}{m} \geq \frac{\alpha(I_{\mathbb{X}(a)} + N - 1)}{N}$$

$\forall \mathrm{a} \in \cap_{m \in \mathbb{N}} U_m$. *In other words Chudnovsky's conjecture holds for a very general set of points.*

Theorem 2.8 (Grifo'18)

If $I^{(hc-h)} \subseteq I^c$ *for some constant* $c \in \mathbb{N}$ *then for all* $r \gg 0$, *we have*

$$I^{(hr-h)} \subseteq I^r.$$

More concretely, this containment holds for all $r \geqslant hc$.

The Bridge: One Containment gives Stable Containment

Theorem 2.9 (–, Grifo, Ha, Nguyễn'20)

Let $I \subseteq \mathbb{K}[\mathbb{P}^N_{\mathbb{K}}]$ be a radical ideal of bigheight h. Let $b \in \mathbb{Z}$. Suppose that for some value $c \in \mathbb{N}$, $I^{(hc-h)} \subseteq \mathfrak{m}^{c(h-b)} I^c$ then for all $r \gg 0$, we have

$$I^{(hr-h)} \subseteq \mathfrak{m}^{r(h-b)} I^r.$$

Theorem 2.10 (–, Grifo, Ha, Nguyễn'20)

Suppose that for some constant $c \in \mathbb{N}$, we have

$$I^{(c(h+m-1)-h+1)} \subseteq \mathfrak{m}^{c(h-1)} \left(I^{(m)}\right)^c.$$

Then for all $t \in \mathbb{N}$,

$$I^{(ct(m+h-1))} \subseteq \mathfrak{m}^{ct(h-1)} (I^{(m)})^{ct}.$$

Example: Fermat Configuration of n^2+3 points

Example 2.11

Let, $I = (x(y^n - z^n), y(z^n - x^n), z(x^n - y^n))$ in $\mathbb{K}[x,y,z]$ where $char\mathbb{K} \neq 2$ containing $n \geqslant 3$ distinct n-th roots of unity; These ideals fail $I^{(3)} \subseteq I^2$. [by Harbourne-Seceleanu'15, also by Dumnicki-Szemberg-Tutaj-Gasińska'13]

But, we can establish the stable Harbourne–Huneke containment for these ideals :

- We prove, $I^{(10)} \subseteq \mathfrak{m}^6 I^6$ i.e, $I^{(2\cdot 6-2)} \subseteq \mathfrak{m}^6 I^6$.
- By Theorem 2.9, $I^{(2r-2)} \subseteq \mathfrak{m}^r I^r$, $r \gg 0$
- So, $I^{(2r-1)} \subseteq \mathfrak{m}^{r-1} I^r$ for $r \gg 0$, and
- $I^{(2r)} \subseteq \mathfrak{m}^r I^r$ for $r \gg 0$.

Example: Böröczky configuration B_{12}

Example 2.12

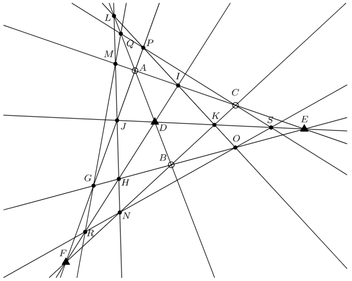

Fig. 1. A configuration of 12 lines with 19 triple points.

Let I be the defining ideal of the Böröczky configuration B_{12} of 19 triple points in $\mathbb{P}^2$,

- Prove $I^{(8)} \subseteq \mathfrak{m}^7 I^5$ i.e., $I^{(2\cdot 5-2)} \subseteq \mathfrak{m}^7 I^5$
- By Theorem 2.9, $I^{(2r-2)} \subseteq \mathfrak{m}^r I^r$, $r \gg 0$
- $I^{(2r-1)} \subseteq \mathfrak{m}^{r-1} I^r$ for $r \gg 0$, and
- $I^{(2r)} \subseteq \mathfrak{m}^r I^r$ for $r \gg 0$.

Result For General Points: Stable Containment & Chudnovsky

Theorem 2.13 (–, Grifo, Ha, Nguyễn'20)

Suppose that char $\mathbb{K} = 0$, $N \geqslant 3$ and I defines the ideal for a general set of s points in $\mathbb{P}^N_{\mathbb{K}}$. If $s \geqslant 4^N$ then the stable containment $I^{(Nr)} \subseteq \mathfrak{m}^{r(N-1)} I^r$, holds for $r \gg 0$.

Consequence of the above theorem:

Theorem 2.14 (–, Grifo, Ha, Nguyễn'20)

Suppose char $\mathbb{K} = 0, N \geq 3$, then Chudnovsky's Conjecture holds for a general set of $s \geq 4^N$ points in $\mathbb{P}^N_{\mathbb{K}}$, that is,

$$\hat{\alpha}(I) \geq \frac{\alpha(I) + N - 1}{N}.$$

Result for General Points: Demailly

Theorem 2.15 (Bisui-Grifo-Ha-Nguyễn)

Suppose that $N \geqslant 3$, $k \geqslant 2m+2$, and $k^N \leqslant s < (k+1)^N$. Let I be the defining ideal of s general points in $\mathbb{P}^N_{\mathbb{K}}$, where char $\mathbb{K} = 0$. We have

$$I^{(c(m+N-1)-N+1)} \subseteq (I^{(m)})^c \mathfrak{m}^{c(N-1)}$$

for some $c \in \mathbb{N}$.

As a consequence of the Theorem we get

Theorem 2.16 (Bisui-Grifo-Ha-Nguyễn)

With the same set up as in Theorem 2.15

$$\widehat{\alpha}(I) \geqslant \frac{\alpha(I^{(m)}) + N - 1}{m + N - 1}.$$

Outline of the proof: Chudnovsky's Conjecture

1. Prove for $s \geq 4^N$ the ideal $I(\mathrm{z})$ of s generic points there is a $c \in \mathbb{N}$ such that
$$I(\mathrm{z})^{(cN-N)} \subset \mathfrak{m}^{r(N-1)} I(\mathrm{z})^c$$
2. So there is an open set W in $\mathbb{A}^{s(N+1)}$ where
$$\pi_{\mathrm{a}}\left(I(\mathrm{z})^{(cN-N)}\right) \subset \mathfrak{m}^{r(N-1)} \pi_{\mathrm{a}}\left(I(\mathrm{z})^c\right)$$
Containment for general set of points i.e,
$$I(\mathrm{a})^{(Nc-N)} \subseteq \mathfrak{m}^{c(N-1)} I(\mathrm{a})^c, \ \mathrm{a} \in W$$
3. Apply Theorem 2.9: For all the points in the open set W we have,
$$I(\mathrm{a})^{(Nr-N)} \subseteq \mathfrak{m}^{r(N-1)} I(\mathrm{a})^r, \quad \text{for all } r \gg 0$$
4. Get,
$$\widehat{\alpha}(I(\mathrm{a})) \geqslant \frac{\alpha(I(\mathrm{a})) + N - 1}{N}.$$

Outline of the proof: Demailly's Conjecture

1. Prove for $s \geq (2m+2)^N$ the ideal $I(\mathrm{z})$ of s generic points there is a $c \in \mathbb{N}$ such that

$$I(\mathrm{z})^{(c(m+N-1)-N+1)} \subset \mathfrak{m}^{r(N-1)}(I(\mathrm{z})^{(m)})^c$$

2. So there is an open set W in $\mathbb{A}^{s(N+1)}$ where

$$\pi_{\mathrm{a}}\left(I(\mathrm{z})^{(c(m+N-1)-N+1)}\right) \subset \mathfrak{m}^{r(N-1)}\pi_{\mathrm{a}}\left((I(\mathrm{z})^{(m)})^c\right)$$

   i.e,

$$I(\mathrm{a})^{(c(m+N-1)-N+1)} \subseteq \mathfrak{m}^{c(N-1)}(I(\mathrm{z})^{(m)})^c, \ \mathrm{a} \in W$$

3. Apply Theorem 2.15: For all the points in the open set W we have,

$$I(\mathrm{a}))^{(ct(m+h-1))} \subseteq \mathfrak{m}^{ct(h-1)}(I(\mathrm{a})^{(m)})^{ct}, \quad \text{for all } t \in \mathbb{N}$$

4. Get,

$$\widehat{\alpha}(I(\mathrm{a})) \geqslant \frac{\alpha(I(\mathrm{a})^{(m)}) + N - 1}{m + N - 1}.$$

Definition 2.17 (Star-Configuration of Co-dim h)

Let $\mathscr{H} = \{H_1, \ldots, H_n\}$ be a collection of $s \geqslant 1$ hypersurfaces in $\mathbb{P}^N_{\mathbb{K}}$. Assume that these hypersurfaces *meet properly*; that is, the intersection of any k of these hypersurfaces either is empty or has codimension k.

$$\mathbb{V}_{h,\mathscr{H}} = \bigcup_{1 \leqslant i_1 < \cdots < i_h \leqslant n} H_{i_1} \cap \cdots \cap H_{i_h}, 1 \leq h \leq N$$

We call $\mathbb{V}_{h,\mathscr{H}}$ a *codimension h star configuration*.

- Let $\mathscr{F} = \{F_1, \ldots, F_n\}$ denotes the forms defining the hyper-planes the defining ideal of $\mathbb{V}_{h,\mathscr{H}}$ is given by

$$I_{h,\mathscr{F}} = \bigcap_{1 \leqslant i_1 < \cdots < i_h \leqslant n} (F_{i_1}, \ldots, F_{i_h}).$$

- (–, Grifo, Ha, Nguyễn'20) $I_{h,\mathscr{F}}^{(r(m+h-1)-h+c)} \subseteq \mathfrak{m}^{(r-1)(h-1)+c-1}(I_{h,\mathscr{F}}^{(m)})^r$.

Determinantal Rings

- For fixed positive integers $t \leqslant \min\{p, q\}$,
- Let X be a $p \times q$ matrix of indeterminates, let $R = \mathbb{K}[X]$,
- let $I = I_t(X)$ denote the ideal of t-minors of X.
- Let $h = (p - t + 1)(q - t + 1)$ be the height of I in R.
- (–, Grifo, Ha, Nguyễn'20) For all $m, r \geqslant 1$, we have

$$I^{(r(h+m-1))} \subseteq \mathfrak{m}^{r(h-1)} \left(I^{(m)}\right)^r.$$

- Demailly like bound holds for determinantal ideals,

$$\widehat{\alpha}(I) \geqslant \frac{\alpha(I^{(m)}) + h - 1}{m + h - 1}.$$

Future Directions

- What happens with the conjectures (Chudnovsky and Harbourne-Huneke) for smaller number of points in $\mathbb{P}^N$?
- Containment of ideals arising from different set up eg. combinatorial, or tropical, or different configurations, or arrangements.
- Demailly's Conjecture generalizes Chudnovsky.
- Fix any $d \in \mathbb{N}$. What is $\dim_{\mathbb{C}}\{f \in \mathscr{L}_{\mathbb{X},m_i} | \deg(f) = d\}$? Let, $\mathbb{X} = \{P_1, \dots, P_s\}$ and $I_{\mathbb{X}} = \mathfrak{p}_1 \cap \cdots \cap p_s$, $\mathbb{Y} = m_1 P_1 + \cdots + m_s P_s \leftrightarrow I_{\mathbb{Y}} = \mathfrak{p}_1^{m_1} \cap \cdots \cap p_s^{m_s}$ (By Zariski-Nagata).

Question 2.1 (Hilbert Functions)

$$\mathscr{H}(I_{\mathbb{Y}}) = \dim_{\mathbb{C}}[I_{\mathbb{Y}}]_d = ?$$

$$\mathscr{H}_{\mathbb{Y}}(d) = \dim_{\mathbb{C}}[\mathbb{C}[X_0, \dots, X_N]/I_{\mathbb{Y}}]_d = ?$$

The SHGH conjecture predicts for maximal Hilbert Function

Thank you