Cohen-Macaulay Type of Weighted Edge Ideals and r-Path Ideals

Shuai Wei

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Outline

Introduction

We investigate the Cohen-Macaulay property of several special classes of monomial ideals that are important for graph theory and combinatorics. Then we compute the Cohen-Macaulay type of these ideals combinatorially.

Let I be a monomial ideal in $R = \mathbb{K}[X_1, \dots, X_d]$.

Definition

Assume I has an irredundant irreducible decomposition $I = \bigcap_{i=1}^{m} J_i$. The Krull dimension of R/I is defined by

$$\dim(R/I) = d - n,$$

where n is the smallest number of generators needed for one of the J_i.

Example

For the irredundant irreducible decomposition ($R = \mathbb{K}[X_1, \dots, X_4]$)

$$I = (X_1X_2, X_2X_3, X_3X_4)R = (X_1, X_3)R \cap (X_2, X_3)R \cap (X_2, X_4)R,$$

the Krull dimension of R/I is $\dim(R/I) = 4 - 2 = 2$.

Fact

$$\min\{i \geq 0 \mid \operatorname{Ext}^i_R(\mathbb{K}, R/I) \neq 0\} \leq \dim(R/I).$$

Definition

Define R/I to be Cohen-Macaulay if

$$\min\{i \geq 0 \mid \operatorname{Ext}^i_R(\mathbb{K}, R/I) \neq 0\} = \dim(R/I),$$

i.e., if $\operatorname{Ext}^i_R(\mathbb{K}, R/I) = 0$ for all $i < \dim(R/I)$.

Remark

$\min\{i \geq 0 \mid \operatorname{Ext}^i_R(\mathbb{K}, R/I) \neq 0\}$ is called the depth of R/I, which can also be defined in terms of regular sequences.

Fact

If R/I is Cohen-Macaulay, then I is unmixed. So Cohen-Macaulayness is unmixed + more. Cohen-Macaylayness is a niceness condition, like being integrally closed. (They're actually related some.)

Let $R = \mathbb{K}[X_1, \ldots, X_d]$ and I a proper monomial ideal in R.

Definition

If R/I is Cohen-Macaulay, define the type of R/I by

$$\operatorname{type}(R/I) = \dim_{\mathbb{K}}(\operatorname{Ext}_R^n(\mathbb{K}, R/I)),$$

where $n = \dim(R/I)$. This measures the complexity of I once you've modded out by a regular sequence.

Definition

R/I is Gorenstein if it is Cohen-Macaulay with type 1.

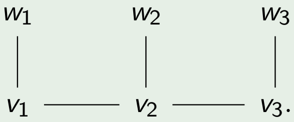
Definition (Villarreal)

Let $G = (V, E)$ be a graph with $V = \{v_1, \ldots, v_d\}$ and $\mathbb{K}$ a field. Let $R = \mathbb{K}[X_1, \ldots, X_d]$. The edge ideal associated to G is the ideal $I(G) \subseteq R$ that is "generated by the edges of G":

$$I(G) = (X_i X_j \mid v_i v_j \in E)R.$$

Example

Consider the following graph G:



The edge ideal of G is

$$I(G) = (X_1X_2, X_2X_3, X_1Y_1, X_2Y_2, X_3Y_3),$$

in $R = \mathbb{K}[X_1, X_2, X_3, Y_1, Y_2, Y_3]$.

Definition

The suspension of a graph $G = (V, E)$ with $V = \{v_1, \ldots, v_d\}$ is the graph ΣG with vertex set

$$V(\Sigma G) = V \sqcup \{w_1, \ldots, w_d\} = \{v_1, \ldots, v_d, w_1, \ldots, w_d\}$$

and edge set

$$E(\Sigma G) = E(G) \sqcup \{v_1 w_1, \ldots, v_d w_d\}.$$

This is also known as the K_1-corona of G.

Theorem (Villarreal)

Let $T = (V, E)$ be a tree with $V = \{v_1, \ldots, v_d\}$ and $R = \mathbb{K}[X_1, \ldots, X_d]$. Then the following conditions are equivalent.

(a) $R/I(T)$ is Cohen-Macaulay,

(b) $I(T)$ is unmixed,

(c) one of the followings holds:

(1) $|V(T)| \leq 2$,

(2) T is a suspension of a tree.

Definition

A vertex cover of $G = (V, E)$ is a subset $V' \subseteq V$ such that for each edge $v_i v_j \in E$ we have $v_i \in V'$ or $v_j \in V'$. A vertex cover V' is minimal if it does not properly contain another vertex cover of G.

Example

The minimal vertex covers for the 2-path

$P_2 = (v_1 —— v_2 —— v_3)$ are

$\textcircled{v_1} —— v_2 —— \textcircled{v_3}$ $\quad$ $v_1 —— \textcircled{v_2} —— v_3$.

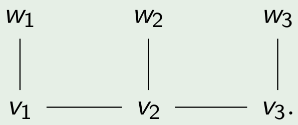
Theorem

If H is a tree such that $R/I(H)$ is Cohen-Macaulay, then $H = \Sigma G$ for some subgraph G by Villarreal. We can compute

$$\mathrm{type}\big(R/I(H)\big) = \sharp\,\{\textit{minimal vertex covers of } G\}.$$

Example

The suspension ΣP_2 of the path $P_2 = (\ v_1 \text{ ——— } v_2 \text{ ——— } v_3\ )$ is



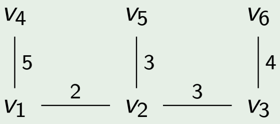
$$\mathrm{type}\big(R/I(\Sigma P_2)\big) = \sharp\,\{\text{minimal vertex covers of } P_2\} = 2.$$

Definition

A weight function on a graph $G = (V, E)$ is a function $\omega : E \to \mathbb{N}$ that assigns a weight to each edge. A weighted graph G_ω is a graph G equipped with a weight function ω.

Example

Let $G = (V, E)$ with $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $E = \{v_1v_2, v_2v_3, v_1v_4, v_2v_5, v_3v_6\}$. We assign a weight to each edge of G, then we get, e.g., the following weighted graph G_ω.



Definition

A weighted suspension of G_ω is a weighted graph $(\Sigma G)_\lambda$ with weight function $\lambda : \Sigma G \to \mathbb{N}$ such that the underlying graph ΣG is a suspension of G and $\lambda(v_i v_j) = \omega(v_i v_j)$ for all $v_i v_j \in E(G)$, i.e., $\lambda|_{E(G)} = \omega$. Graphically, $(\Sigma G)_\lambda$ has the form

$$\begin{array}{ccccccccc} \cdots & & w_i & & w_j & & w_k & & \cdots \\ & & \Big| {\scriptstyle \lambda(v_i w_i)} & & \Big| {\scriptstyle \lambda(v_j w_j)} & & \Big| {\scriptstyle \lambda(v_k w_k)} & & \\ \cdots & \text{———} & v_i & \underset{\omega(v_i v_j)}{\text{———}} & v_j & \underset{\omega(v_j v_k)}{\text{———}} & v_k & \text{———} & \cdots . \end{array}$$

Definition

A weighted vertex cover of G_ω is an ordered pair $(V', \delta') \in \Omega$ such that the set V' is a vertex cover of G and for each edge $v_iv_j \in E$, we have

(a) $v_i \in V'$ and $\delta'(v_i) \leq \omega(v_iv_j)$, or

(b) $v_j \in V'$ and $\delta'(v_j) \leq \omega(v_iv_j)$.

The number $\delta'(v_i)$ is the *weight* of v_i.

Definition

Given two weighted vertex covers (V_1', δ_1') and (V_2', δ_2') of G_ω, we write $(V_2', \delta_2') \leq (V_1', \delta_1')$ if $V_2' \subseteq V_1'$ and $\delta_2'(v_i) \geq \delta_1'(v_i)$ for all $v_i \in V_2'$. A weighted vertex cover (V', δ') is minimal if there does not exist another weighted vertex cover (V'', δ'') such that $(V'', \delta'') < (V', \delta')$. We define $|(V', \delta')| = |V'|$.

Example

The minimal weighted vertex covers for the weighted 2-path $(P_2)_\omega = (\ v_1 \overset{2}{\text{——}} v_2 \overset{3}{\text{——}} v_3)$ are

$$\boxed{v_1^2} \overset{2}{\text{——}} v_2 \overset{3}{\text{——}} \boxed{v_3^3} \qquad v_1 \overset{2}{\text{——}} \boxed{v_2^2} \overset{3}{\text{——}} v_3.$$

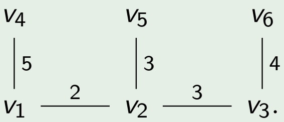
Definition (Chelsey Paulsen and Keri Sather-Wagstaff [2])

The weighted edge ideal associated to G_ω is the ideal $I(G_\omega) \subseteq R$ that is "generated by the weighted edges of G":

$$I(G_\omega) = \left(X_i^{\omega(v_iv_j)} X_j^{\omega(v_iv_j)} \mid v_iv_j \in E\right)R.$$

Example

Consider the following graph H_λ:



$$I(H_\lambda) = (X_1^2X_2^2, X_2^3X_3^3, X_1^5X_4^5, X_2^3X_5^3, X_3^4X_5^4)R,$$

$$R = \mathbb{K}[X_1, X_2, X_3, X_4, X_5, X_6].$$

Theorem (Chelsey Paulsen and Keri Sather-Wagstaff [2])

Let H_λ be a weighted tree with $V = \{v_1, \ldots, v_d\}$ and $R = \mathbb{K}[X_1, \ldots, X_d]$. Then the following conditions are equivalent.

(a) $R/I(H_\lambda)$ is Cohen-Macaulay,

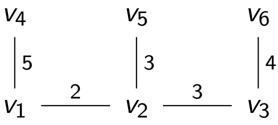
(b) $I(H_\lambda)$ is unmixed,

(c) one of the following holds:

(1) $|V(H_\lambda)| \leq 2$ or

(2) H_λ is a weighted suspension of a weighted tree G_ω such that $\lambda(v_iv_j) \leq \lambda(v_iw_i)$ and $\lambda(v_iv_j) \leq \lambda(v_jw_j)$ for each $v_iv_j \in E(G)$. We write $H_\lambda = (\Sigma G)_\lambda$.

For example, H_λ we just saw before satisfies the condition (c).(2).



Theorem

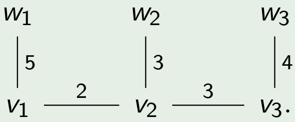
Let H_λ be a weighted tree such that $R/I(H_\lambda)$ is Cohen-Macaulay. Then $H_\lambda = (\Sigma G)_\lambda$ for some weighted subtree G_ω such that $\lambda(v_i v_j) \leq \lambda(v_i w_i)$ and $\lambda(v_i v_j) \leq \lambda(w_j v_j)$ for each $v_i v_j \in E(G)$ by Chelsey-KW. We can compute

$$\operatorname{type}\left(\frac{R}{I(H_\lambda)}\right) = \sharp\{\textit{minimal weighted vertex covers of } G_\omega\}.$$

Example

The weighted suspension $(\Sigma P_2)_\lambda$ of the weighted path

$(P_2)_\omega = \left(v_1 \overset{2}{\text{———}} v_2 \overset{3}{\text{———}} v_3 \right)$ is



$$\begin{aligned} \mathrm{type}\big(R/I((\Sigma P_2)_\lambda)\big) &= \sharp\,\{\text{minimal weighted vertex covers of } (P_2)_\omega\} \\ &= 2. \end{aligned}$$

We can generalize all of this to the r-path setting. The Cohen-Macaulay property is characterized for path ideals of trees and some weighted path ideals for weighted trees by Morey, et al. and Kubik-SW. We have a complete characterization subsuming the previous partial characterizations. We can also compute the Cohen-Macaulay type for all of these Cohen-Macaulay ideals.

Definition

An r-path vertex cover of G is a subset $V' \subseteq V$ s.t. for any r-path $v_{i_1} \cdots v_{i_{r+1}}$ in G, we have $v_{i_j} \in V'$ for some $j \in \{1, \ldots, r+1\}$. An r-path vertex cover V' is minimal if it does not properly contain another r-path vertex cover of G.

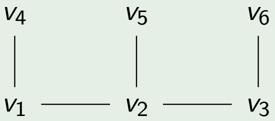
Definition (Conca and De Negri)

Let $G = (V, E)$ with $V = \{v_1, \dots, v_d\}$. Let $R = \mathbb{K}[X_1, \dots, X_d]$. The r-path ideal associated to G is the ideal $I_r(G) \subseteq R$ that is "generated by the paths in G of length r":

$$I_r(G) = (X_{i_1} \dots X_{i_{r+1}} \mid v_{i_1} \dots v_{i_{r+1}} \text{ is a path in } G)R.$$

Example

Consider the following graph G in $R = \mathbb{K}[X_1, \dots, X_6]$.



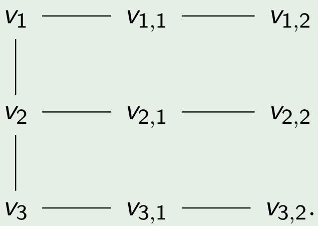
$$I_2(G) = (X_4X_1X_2, X_1X_2X_5, X_1X_2X_3, X_5X_2X_3, X_2X_3X_6)\, R.$$

Definition

The *r-path suspension* of a graph G is the graph $\Sigma_r G$ obtained by adding a new path of length r to each vertex of G. The new r-paths are called *r-whiskers*.

Example

The 2-path suspension $\Sigma_2 P_2$ of the 2-path $G = P_2 = (\ v_1 \ \text{———}\ v_2 \ \text{———}\ v_3\ )$ is



Definition

Define $q : V(\Sigma_{r-1}G) \to V(G)$ as $q(v_{i,j}) = v_i$. Let $V'' \subseteq V(\Sigma_{r-1}G)$. Then $q(V'') = \{v_i \mid \exists\ v_{i,j} \in V''\}$ and we set

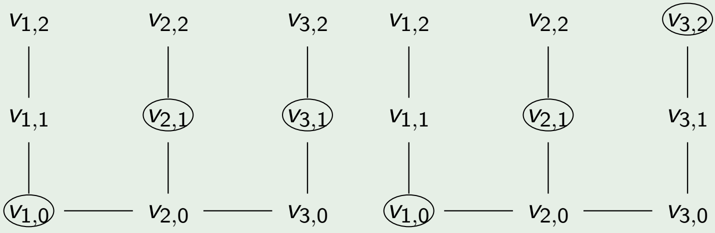
$$\begin{aligned} \gamma_{V''} : q(V'') &\to \mathbb{N} \\ v_i &\mapsto 1 + \min\{j \mid v_{i,j} \in V''\}. \end{aligned}$$

Definition

Given two minimal r-path vertex covers V_1', V_2' of $\Sigma_{r-1}G$. Write $(V_1', \gamma_{V_1'}) \leq_{\not p} (V_2', \gamma_{V_2'})$ if $q(V_1') \subseteq q(V_2')$ and $\gamma_{V_1'}|_{q(V_1')} \geq \gamma_{V_2'}|_{q(V_1')}$. A minimal r-path vertex cover $(V', \gamma_{V'})$ is $\underline{\not p\text{-minimal}}$ if there is not another r-path vertex cover $(W', \gamma_{W'})$ such that $(W', \gamma_{W'}) <_{\not p} (V', \gamma_{V'})$.

Example

The following are two minimal 3-path vertex covers of $\Sigma_2 P_2$.



$$V_1'' = \{v_{1,0}, v_{2,1}, v_{3,1}\}$$
$$q(V_1'') = \{v_1, v_2, v_3\}$$
$$(V_1'', \gamma_{V_1''}) = \{v_1^1, v_2^2, v_3^2\}$$

$$V_2'' = \{v_{1,0}, v_{2,1}, v_{3,2}\}$$
$$q(V_2'') = \{v_1, v_2, v_3\}$$
$$\underbrace{(V_2'', \gamma_{V_2''}) = \{v_1^1, v_2^2, v_3^3\}}_{\wp\text{-minimal}}$$

Definition

Let v_i be a vertex of degree 1 in G that is not a part of any r-path in G. We write that v_i is an r-pathless leaf of G.

Fact

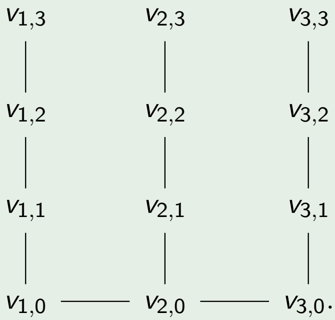
If G is a tree and $R/I_r(G)$ is Cohen-Macaulay, then there exists a subtree H such that $\Sigma_r H$ is obtained by pruning a sequence of r-pathless leaves from G.

Theorem

$$\text{type}\left(\frac{R'}{I_r(\Sigma_r G)}\right) = \sharp\{\wp\text{-minimal } r\text{-path vertex covers of } \Sigma_{r-1} G\}.$$

Example

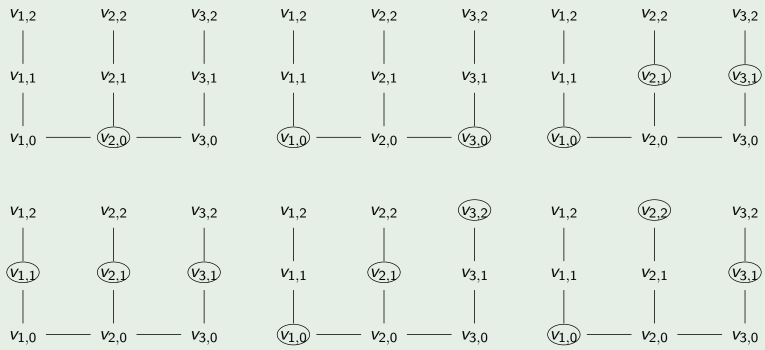
The 3-path suspension $\Sigma_3 P_2$ of $P_2 = (\; v_1 \;\text{———}\; v_2 \;\text{———}\; v_3 \;)$ is



We depict the minimal 3-path vertex covers of $\Sigma_2 P_2$ in the following sketches.

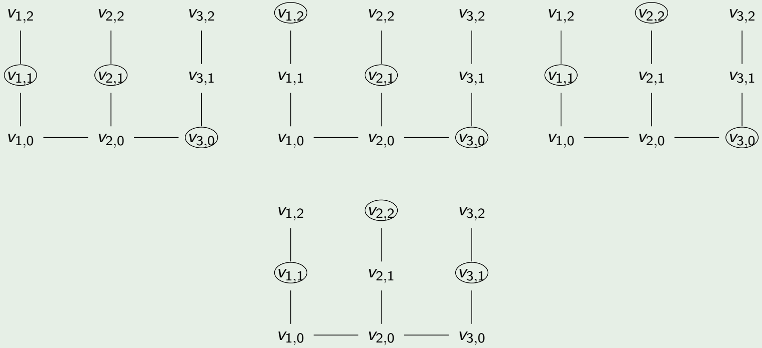
Example (Continued)

(The first 6 minimal 3-path vertex covers of $\Sigma_2 P_2$.)



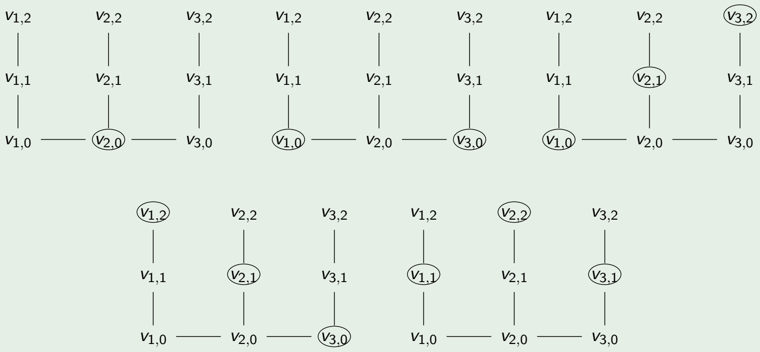
Example (Continued)

(The last 4 minimal 3-path vertex covers of $\Sigma_2 P_2$.)



Example (Continued)

The $\wp$-minimal 3-path vertex covers of $\Sigma_2 P_2$ are the following.



Hence

$$\text{type}(R'/I_3(\Sigma_3 P_2)) = 5.$$

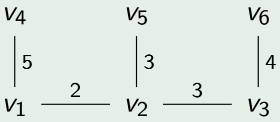
Definition (Bethany Kubik and Keri Sather-Wagstaff [1])

Let $G = (V, E)$ with $V = \{v_1, \ldots, v_d\}$. Let $R = \mathbb{K}[X_1, \ldots, X_d]$. The **weighted r-path ideal** associated to G_ω is the ideal $I_r(G_\omega) \subseteq R$ that is "generated by the max-weighted paths in G of length r":

$$I_r(G_\omega) = \left(X_{i_1}^{e_{i_1}} \ldots X_{i_{r+1}}^{e_{i_{r+1}}} \;\middle|\; \begin{array}{l} v_{i_1} \ldots v_{i_{r+1}} \text{ is a path in } G \text{ with} \\ e_{i_1} = \omega(v_{i_1} v_{i_2}), \\ e_{i_j} = \max\{\omega(v_{i_{j-1}} v_{i_j}), \omega(v_{i_j}, v_{i_{j+1}})\} \\ \text{for } 1 < j \leq r \\ \text{and } e_{i_{r+1}} = \omega(v_{i_r} v_{i_{r+1}}) \end{array} \right) R.$$

Example

Consider the following graph G_ω in $R = \mathbb{K}[X_1, \ldots, X_6]$.



The weighted 2-path ideal of G_ω is

$$I_2(G_\omega) = (X_4^5X_1^5X_2^2, X_1^2X_2^3X_5^3, X_1^2X_2^3X_3^3, X_5^3X_2^3X_3^3, X_2^3X_3^4X_6^4)R.$$

Definition

A weighted r-path suspension of G_ω is a weighted graph $(\Sigma_r G)_\lambda$ with weight function $\lambda : \Sigma_r G \to \mathbb{N}$ such that the underlying graph $\Sigma_r G$ is a r-path suspension of G and $\lambda|_{E(G)} = \omega$.

Example

$(\Sigma_2 P_2)_\lambda$ of $(P_2)_\omega = \left(\; v_1 \overset{1}{\text{———}} v_2 \overset{2}{\text{———}} v_3 \; \right)$ is

$$
\begin{array}{ccccc}
v_1 & \overset{4}{\text{———}} & v_{1,1} & \overset{3}{\text{———}} & v_{1,2} \\
1 \big| & & & & \\
v_2 & \overset{3}{\text{———}} & v_{2,1} & \overset{3}{\text{———}} & v_{2,2} \\
2 \big| & & & & \\
v_3 & \overset{2}{\text{———}} & v_{3,1} & \overset{5}{\text{———}} & v_{3,2}.
\end{array}
$$

We have similar definitions for minimal weighted vertex cover of $(\Sigma_r G)_\lambda$, $\wp$-minimal weighted r-path vertex cover of $(\Sigma_{r-1} G)_{\lambda'}$, and have a similar combinatorial formula to compute the type of $\frac{R'}{I_r((\Sigma_r G)_\lambda)}$.

Theorem

$$\text{type}\left(\frac{R'}{I_r((\Sigma_r G))_\lambda}\right) = \sharp\left\{\wp\textit{-minimal weighted r-path vertex covers}\right.$$
$$\left.\textit{of } (\Sigma_{r-1} G)_{\lambda'}, \ \textit{with } \lambda' = \lambda|_{\Sigma_{r-1} G}\right\}.$$

Thank You!

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