Newton-Okounkov Bodies, Rees Algebras and, Analytic Spread of Graded Families of Monomial Ideals

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CHAMPS

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Investigate the connection between algebraic properties and invariant of a graded family of ideals, and geometric properties and invariant of an associated convex body.

Question

For a graded family of monomial ideals, use combinatorial data of convex bodies to understand :

1. Noetherian property of the Rees algebra.
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1. Noetherian property of the Rees algebra.
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Rees Algebra and Newton-Okounkov body

Notation and Terminology :

1. $\mathbf{k}$ is a field, $R = \mathbf{k}[x_1, \ldots, x_n]$, and $\mathfrak{m} = (x_1, \ldots, x_n)$.
2. $\mathcal{I} = \{I_k\}_{k \in \mathbb{N}}$ is a *graded* family of ideals in R if, for all $p, q \in \mathbb{N}$, $I_p I_q \subseteq I_{p+q}$.

Definition

Let $\mathcal{I} = \{I_k\}_{k \in \mathbb{N}}$ be a graded family of monomial ideals in R.

1. The *Rees algebra* of $\mathcal{I}$:

$$\mathcal{R}(\mathcal{I}) := R \oplus I_1 t \oplus I_2 t^2 \oplus \cdots \subseteq R[t].$$

2. The *Newton-Okounkov body* of $\mathcal{I}$ is defined to be

$$\Delta(\mathcal{I}) = \overline{\bigcup_{k \in \mathbb{N}} \left\{ \frac{\mathbf{a}}{k} \;\middle|\; x^{\mathbf{a}} \in I_k \right\}} \subseteq \mathbb{R}^n.$$

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- Systematically developed by Lazarsfeld and Mustaţă (2009), and by Kaveh and Khovanskii (2012–2014).
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Newton-Okounkov body

Example

1. if $\mathcal{I} = \{I^k\}_{k\in\mathbb{N}}$ is the family of ordinary powers of a monomial ideal I then,

$$\Delta(\mathcal{I}) = \mathrm{NP}(I) := \text{convex hull}\left(\left\{\mathbf{a}\in\mathbb{N}^n \;\middle|\; x^{\mathbf{a}}\in I\right\}\right),$$

which is the *Newton polyhedron* of I.

2. if $\mathcal{I} = \{I^{(k)}\}_{k\in\mathbb{N}}$ is the family of symbolic powers of a monomial ideal I then,

$$\Delta(\mathcal{I}) = \mathrm{SP}(I) := \bigcap_{\mathfrak{p}\in\mathrm{maxAss}(I)} \mathrm{NP}(Q_{\subseteq\mathfrak{p}}).$$

maxAss(I) denote the set of maximal associated primes of I and $Q_{\mathfrak{p}} = R\cap IR_{\mathfrak{p}}$ for $\mathfrak{p}\in\mathrm{maxAss}(I)$. This is the *symbolic polyhedron* of I, introduced by Cooper, Embree, Hà, Hoefel.

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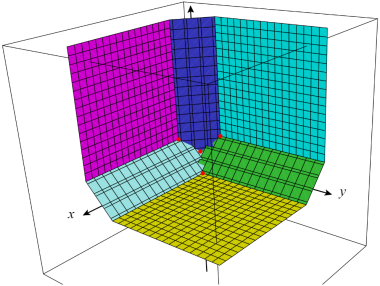

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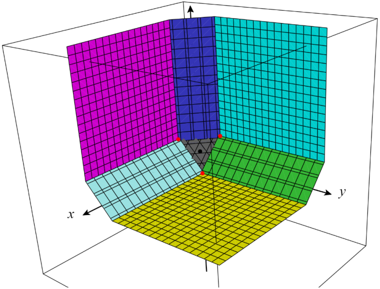

The Newton polyhedron of I.

Newton-Okounkov body

Remark

It can be shown that $\Delta(\mathcal{I})$ is a convex set. Thus, it follows from the definition that

$$\Delta(\mathcal{I}) = \overline{\bigcup_{k\in\mathbb{N}} \frac{1}{k}\mathrm{NP}(I_k)}.$$

In particular, $\Delta(\mathcal{I})$ is the closure of the *limiting body*

$$\mathcal{C}(\mathcal{I}) := \bigcup_{k\in\mathbb{N}} \frac{1}{k}\mathrm{NP}(I_k),$$

of $\mathcal{I}$. This object was named and investigated recently by Camarneiro et. al (2021) ; the same asymptotic object was also studied by Mustata (2002), Wolfe (2008), Mayes (2012) in different contexts.

Noetherian property of Rees algebras

1. Determining when the Rees algebra of a graded family of ideals is Noetherian is a difficult problem.
2. Many examples existed to show that the Rees algebra of the family of symbolic powers of an ideal needs not be Noetherian (Cutkosky (1991), Huneke (1982) and Roberts (1985)).
3. The symbolic Rees algebra of a monomial ideal is known to be Noetherian (cf. Herzog-Hibi-Trung (2007)).

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Focus : the Rees algebra of a graded family of monomial ideals.

Lemma

Let $\mathcal{I} = \{I_k\}_{k\in\mathbb{N}}$ and $\mathcal{J}$ be graded families of monomial ideals such that $\overline{\mathcal{I}} = \overline{\mathcal{J}}$, where $\overline{\mathcal{I}}$ denote the graded family $\{\overline{I_k}\}_{k\in\mathbb{N}}$. Then,

$$\Delta(\mathcal{I}) = \Delta(\mathcal{J}).$$

Theorem (Hà–Nguyễn)

Let $\mathcal{I} = \{I_k\}_{k\in\mathbb{N}}$ be a graded family of monomial ideals in R. TFAE :

1. *$\mathcal{R}(\mathcal{I})$ is Noetherian.*
2. *$\mathcal{R}(\overline{\mathcal{I}})$ is Noetherian.*
3. *$\mathcal{C}(\mathcal{I})$ is a polyhedron.*
4. *There exists an integer c such that $\Delta(\mathcal{I}) = \dfrac{1}{c}\mathrm{NP}(I_c)$.*

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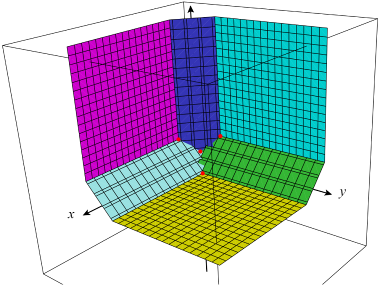
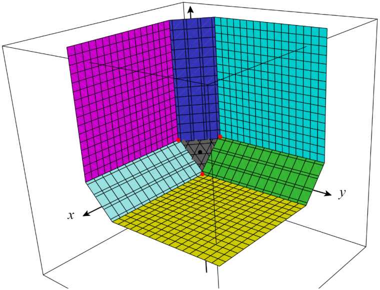
Example

Let $I = (xy, yz, zx) = (x, y) \cap (y, z) \cap (z, x)$. Then

$\mathrm{SP}(I) = \frac{1}{2}\,\mathrm{NP}(I^{(2)})$ and $\frac{1}{2}\,\mathrm{NP}(I^{(2)}) = \frac{1}{2k}\,\mathrm{NP}(I^{(2k)})$ for all k.





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Let $I = (ab, bc, cd, de, ea, fa, fb, fc, fd, fe) \subseteq \mathbf{k}[a, \dots, f]$ be the edge ideal of a cone over a 5-cycle. Then, $\mathrm{SP}(I)$ has 17 vertices (written as the columns of the following matrix) :

$$\begin{pmatrix}
1/5 & 1/3 & 1/2 & 0 & 0 & 0 & 1/2 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
1/5 & 1/3 & 1/2 & 1/2 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
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\end{pmatrix}.$$

In this example, $\mathrm{NP}(I^{(30)}) = 30\,\mathrm{SP}(I)$, and $c = 30$ is the least possible integer for $\mathrm{NP}(I^{(c)}) = c\,\mathrm{SP}(I)$ to hold.

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For a monomial ideal I and a real number $r \geq 0$, the *r-th real power* of I is

$$\overline{I^r} := \{x^{\mathbf{a}} \mid \mathbf{a} \in r\,\mathrm{NP}(I) \cap \mathbb{N}^n\}.$$

Let $f : \mathbb{N} \to \mathbb{R}_{\geq 0}$ be a *subadditive* function ; i.e, $f(m) + f(n) \geq f(m+n), \forall m, n \in \mathbb{N}$; and suppose that $\lim_{k\to\infty} \frac{f(k)}{k} \in \mathbb{Q}$ and is attained at some value k_0. Consider

$$\mathcal{I} = \{I_k\}_{k\in\mathbb{N}}, \quad I_k = \overline{I^{f(k)}}.$$

Then

$$\mathcal{C}(\mathcal{I}) = \bigcup_{k\in\mathbb{N}} \frac{1}{k}\,\mathrm{NP}(I_k) = \frac{f(k_0)}{k_0}\,\mathrm{NP}(I),$$

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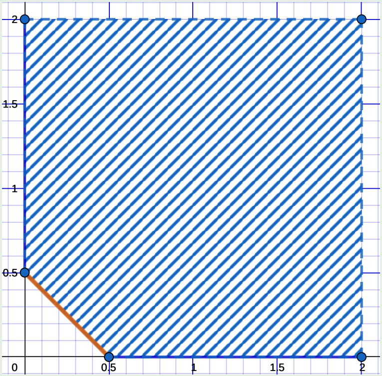
A Non-Example

Example

Let $R = \mathbf{k}[x, y]$ and consider the graded family $\mathcal{I} = \{I_k\}_{k \in \mathbb{N}}$ with

$$I_k = (x, y)^{\lceil k/2 \rceil + 1} \subseteq R.$$



- $\Delta(\mathcal{I})$ is the blue region including the red segment
- $C(\mathcal{I})$ is the blue region only, which fails to be a polyhedron, hence, $\mathcal{R}(\mathcal{I})$ is not Noetherian.

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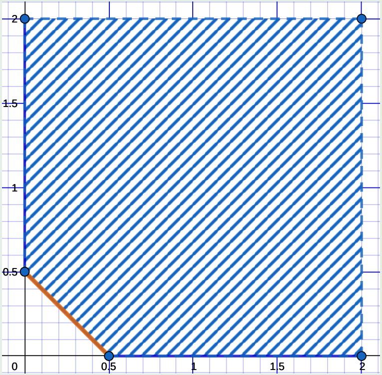

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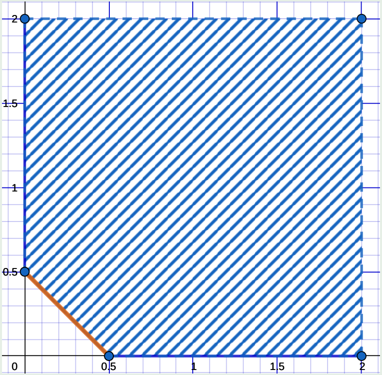

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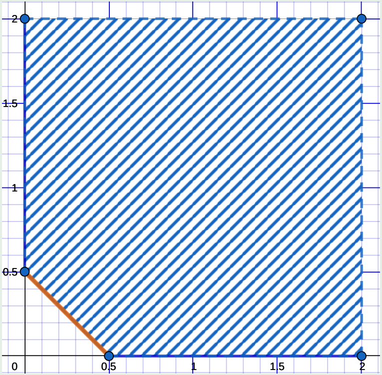

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Remark

For any non-empty closed convex set $P \subseteq \mathbb{R}^n_{\geq 0}$ absorbing $\mathbb{R}^n_{\geq 0}$, i.e., $P + \mathbb{R}^n_{\geq 0} \subseteq P$, there exists a graded family of monomial ideals $\mathcal{I}$ such that $\Delta(\mathcal{I}) = P$. In particular, $\mathcal{I} = \{I_k\}_{k \in \mathbb{N}}$ where

$$I_k = \langle \{ x^{\mathbf{a}} \mid \mathbf{a} \in kP \cap \mathbb{Z}^n \} \rangle.$$

For graded family $\mathcal{I} = \{I_k\}_{k\in\mathbb{N}}$ of monomial ideals,

- The formula "*multiplicity = volume*" :

$$e(\mathcal{I}) := n! \lim_{k\to\infty} \frac{\dim(R/I_k)}{k^n} = n!\,\mathrm{covol}(\Delta(\mathcal{I})).$$

  where $\mathrm{covol}(\Delta(\mathcal{I}))$ is the volume of the complement of $\Delta(\mathcal{I})$. (Mustata 2002, Kaveh-Khovanskii 2014)

- Study *containment problem* of a monomial ideal I through SP(I) (Cooper *et. al.* 2016).
- Study the *asymptotic resurgence numbers* through skew valuation on NP(I) (Dipasquale *et. al.* 2018)
- The Waldschmidt constant

$$\hat{\alpha}(\mathcal{I}) := \lim_{k\to\infty} \frac{\alpha(I_k)}{k} = \min\{v_1+\cdots+v_n \mid (v_1,\ldots,v_n)\in\Delta(\mathcal{I})\}.$$

  where $\alpha(I)$ denote the minimal degree of generators of I (Cooper *et. al.* 2016, Camarneiro *et. al.* 2021).

Newton-Okounkov Body and Algebraic Invariant

For graded family $\mathcal{I} = \{I_k\}_{k\in\mathbb{N}}$ of monomial ideals,

- The formula "*multiplicity = volume*" :

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  where $\mathrm{covol}(\Delta(\mathcal{I}))$ is the volume of the complement of $\Delta(\mathcal{I})$. (Mustata 2002, Kaveh-Khovanskii 2014)
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Analytic Spread

Definition

Let $\mathcal{I} = \{I_k\}_{k\in\mathbb{N}}$ be a graded family of monomial ideals in R. The *analytic spread* of $\mathcal{I}$ is defined to be :

$$\ell(\mathcal{I}) := \dim \mathcal{R}(\mathcal{I})/\mathfrak{m}\mathcal{R}(\mathcal{I}).$$

- When $\mathcal{I}$ is a *filtration*, $\ell(\mathcal{I}) = \dim \mathcal{R}(\mathcal{I})/\mathfrak{m}\mathcal{R}(\mathcal{I})$ exists and is bounded above by $\dim R$ (Cutkosky–Sarkar (2021)).
- It turns out that $\ell(\mathcal{I}) = \dim \mathcal{R}(\mathcal{I})/\mathfrak{m}\mathcal{R}(\mathcal{I})$ exists and is bounded above by $\max\{\ell(I_k)\}$ when $\mathcal{I}$ is a graded family (Hà–Nguyễn).
- When $\mathcal{I} = \{I^k\}_{k\in\mathbb{N}}$ or $\mathcal{I} = \{I^{(k)}\}_{k\in\mathbb{N}}$, we use $\ell(I)$ or $\ell_s(I)$ for $\ell(\mathcal{I})$.

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Analytic spread

For a polyhedron $P \subseteq \mathbb{R}^n$, let $\mathtt{mdc}(P)$ denote the *maximum dimension of a compact face* in P.

Theorem (Hà–Nguyễn)

Let $\mathcal{I}$ be a Noetherian graded family of monomial ideals in R. Then,

$$\ell(\mathcal{I}) = \mathtt{mdc}(\Delta(\mathcal{I})) + 1.$$

Question

Do we have the same result when $\Delta(\mathcal{I})$ is a rational polyhedron ?

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Corollary

Let $\mathcal{I}$ and $\mathcal{J}$ be Noetherian graded families of monomial ideals such that $\overline{\mathcal{I}} = \overline{\mathcal{J}}$. Then,

$$\ell(\mathcal{I}) = \ell(\mathcal{J}).$$

Corollary

Let I be a monomial ideal.

1. $\ell(I) = \mathrm{mdc}(\mathrm{NP}(I)) + 1$ *(Bivià-Ausina (2003)).*
2. $\ell_s(I) = \mathrm{mdc}(\mathrm{SP}(I)) + 1.$

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Example

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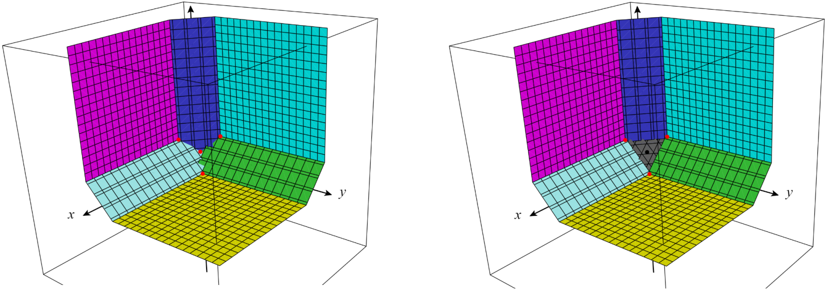
Let $I = (xy, yz, zx) = (x, y) \cap (y, z) \cap (z, x)$. Then

$$\ell(I) = 3 \text{ and } \ell_s(I) = 2.$$



Analytic Spread and Growth of Numbers of Minimal Generators

Definition

$$\ell^*(\mathcal{I}) = \min\{t \in \mathbb{R} \mid \mu(I_k) = O(k^{t-1})\}.$$

- If $\mathcal{I}$ is Noetherian, we have $\ell^*(\mathcal{I}) = \ell(\mathcal{I})$.
- It is not known in general if $\ell^*(\mathcal{I})$ is always finite.
- $\ell^*(\mathcal{I})$ has been investigated by various authors (Hoa–Kimura–Terai–Trung (2017), Dao–Montaño (2020)).

Example

Let $\mathcal{I} = \{I_k\}_{k\in\mathbb{N}}$ be a graded family of $\mathfrak{m}$-primary, $\mathfrak{m}$-full monomial ideals (e.g, integrally closed). Suppose that $\Delta(\mathcal{I})$ is a rational polyhedron. Then,

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Generation Type and Veronese Degree

Fact

The symbolic Rees algebra $\mathcal{R}_s(I) = \bigoplus_{n \in \mathbb{N}} I^{(n)} t^n$ *is finitely generated iff there exists* d*, such that* $I^{(dk)} = (I^{(d)})^k \forall k \geq 1$.

Definition

The standard Veronese degree of an ideal I is defined to be

$$\operatorname{svd}(I) := \inf\{d \mid I^{(dk)} = (I^{(d)})^k \text{ for all } k \geq 1\}.$$

Theorem (Hà–Nguyễn)

Let I *be a squarefree monomial ideal and suppose that* $\{v_1, \ldots, v_r\}$ *are the vertices of* $\mathrm{SP}(I)$*. Let* c *be the least common multiple of the denominators appearing in the coordinates of* $v_1, \ldots, v_r$*. Then*

$$c \leq \operatorname{svd}(I) \leq (\ell_s(I) - 1)c.$$

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The *symbolic generation type* of I is defined to be the maximum generating degree of $\mathcal{R}_s(I)$. That is,

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Let I be a squarefree monomial ideal. Suppose that $\{v_1, \ldots, v_r\}$ are the vertices of $\mathrm{SP}(I)$. Let d_i be the least common multiple of the denominator of coordinates of v_i, for $i = 1, \ldots, r$. Set $D = \max\{d_1, \ldots, d_r\}$.

1. *If $D \geq 2$, then $\operatorname{sgt}(I) \leq \max\{\ell_s(I)D - 1, D\}$.*
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Thank you for listening !