F-purity Deforms in $\mathbb{Q}$-Gorenstein Rings

Joint work with A. Simpson (UIC)

Assumptions: $R = (R, m, k)$ is a Commutative Noetherian local ring with identity.

Deformation Problem: Suppose $\mathcal{P}$ is a property of rings, regular, Gor., C-M, complete, if $\exists$ NZD $f \in R$ s.That $R/\langle f \rangle$ is $\mathcal{P}$, is R also $\mathcal{P}$? If yes, Then $\mathcal{P}$ deforms.

Examples: Reg., Gor., C-M

Non-Example: Complete.

Deformation Problems in Prime Characteristic

Assumptions: R is of prime char $p > 0$

$\forall e \in \mathbb{N}$ $\quad F^e : R \longrightarrow R$, $\quad r \longmapsto r^{p^e}$ $\qquad (r+s)^{p^e} = r^{p^e} + s^{p^e}$

$\boxed{\varphi : S \to T}$

If $M \in \mathrm{Mod}(R)$ let $F^e_* M \in \mathrm{Mod}(R)$ be The module obtained from M via restriction of scalars under F^e.

If $m \in F^e_* M$, $r \in R$, $r \cdot m = F^e(r) m = r^{p^e} m$.

Assumption: R is F-finite if $M \in \mathrm{mod}(R) \Rightarrow F^e_* M \in \mathrm{mod}(R)$

Example: $R = \mathbb{F}_p[[x_1, \ldots, x_d]]$, $F^e_* R \cong R^{\oplus p^{ed}}$ with R-basis $\{x_1^{i_1} \cdots x_d^{i_d} \mid 0 \le i_j \le p^e - 1\}$.

Theorem (Kunz 60's) $F^e_* R$ is free $\Leftrightarrow$ R is regular.

Prime Char Singularities:

1. F-regular: $F^e_* R$ has "lots" of free summands
2. F-pure: $F^e_* R$ has at least 1 free summand
3. F-injective: $\boxed{R \xrightarrow{F^e} R}$ induces 1-1 maps $\to H^i_m(R) \xrightarrow{F^e} H^i_m(R)$

$① \Rightarrow ② \Rightarrow ③$, $\quad ③ \Rightarrow ②$ if R is Gor.
Fedder 83'

~~Deformat~~
When Do ①, ② & ③ deform?

Fedder 83':

✓ • F-injective deforms if C-M (Easy)
  - Very hard in non-C-M rings (still open)
  - F-purity deforms in Gorenstein rings

!!! • F-purity does not deform, but the example is not a domain

Late 90's & Early 00's:

!!! • Singh $\exists$ 3-dim'l C-M normal R which is not ① F-reg ② F-pure ③ $\mathbb{Q}$-Gor. but $\exists$ NZD $f \in R$: $R/\langle f \rangle$ is ①, $\Rightarrow$ ② $\overset{\text{Ig dim 2}}{\Rightarrow}$, ③

✓ • Aberbach, Katzman, MacCrimmon: F-regularity deforms in $\mathbb{Q}$-Gor. rings.

✓ • Hara, Watanabe: F-purity deforms in $\mathbb{Q}$-Gor. rings whose index is not divisible by the char.

2 Conjectures:
1. F-purity deforms in $\mathbb{Q}$-Gor. rings.
2. F-injectivity deforms.

Theorem (–, Simpson) Conjecture ① is OK.

$\mathbb{Q}$-Gorenstein: Assume R is normal ((S_2) & (G_1)) — Gor. in codim 1

$\omega_R \cong J \subsetneq R$, J is an ideal of pure ht 1.
canonical module

R is $\mathbb{Q}$-Gor if $\exists N \in \mathbb{N}$ such That $J^{(N)} = \langle a \rangle \cong R$. The index of R is The least N such That $J^{(N)} \cong R$.

- In a C-M ring $\mathbb{Q}$-Gor. of index 1 = Gor.

What is our attack?

(R, m, k) $\mathbb{Q}$-Gor., $\exists$ NZD f s.That $R/\langle f \rangle$ is $\mathbb{Q}$-Gor. & F-pure.

Choose $\omega_R \cong J \subsetneq R$ so That f is a NZD on R/J. (Make ht 1 comp. of J do not contain f).

Assume The index is Np^e where $p \nmid N$.

Let $I = J^{(N)}$, $\quad I^{(p^e)} \cong J^{(Np^e)} = \langle a \rangle \cong R$.

$$R \longrightarrow S := \frac{R \oplus It \oplus I^{(2)} t^2 \oplus \cdots}{\langle I^{(p^e)} t^{p^e} - 1 \rangle}$$

cyclic cover of R. $\qquad J^{(Np^e)} \cong R$, $p \nmid N$

• R is F-pure $\Leftrightarrow$ S is F-pure!
Carvajal-Rojas, Stäbler

• S is $\mathbb{Q}$-Gor. of index N
- Enough to show S/fS is F-pure!

$$\begin{tikzcd} R \arrow[r] \arrow[d] & S \arrow[d] \\ R/\langle f \rangle \arrow[r] & S/fS \end{tikzcd}$$

We show S/fS is a cyclic of $R/\langle f \rangle$!

• $R/\langle f \rangle$ is F-pure $\Rightarrow$ S/fS is F-pure $\overset{\text{b/c index rel. prime to } p}{\Rightarrow}$ S is F-pure $\Rightarrow$ R is F-pure.

How is S/fS a cyclic cover of $R/\langle f \rangle$?

$\forall t \in \mathbb{N}$
$$\frac{I^{(t)}}{f I^{(t)}} \cong \frac{(I^{(t)}, f)}{(f)} = \left(\frac{(I, f)}{(f)}\right)^{(t)}$$ which shows S/fS is a cyclic cover of $R/\langle f \rangle$.

Why is natural to expect $\dfrac{I^{(t)}, f}{(f)} = \left(\dfrac{(I, f)}{(f)}\right)^{(t)}$?

• We expect R to be F-pure,
- for simplicity assume R is C-M.

R is C-M $\Rightarrow$ $F^e_* R$ is a C-M R-module
$\Rightarrow$ all direct summands of $F^e_* R$ are also C-M.

We expect $F^e_* R \cong R \oplus$ —

• Recall $I^{(p^e)} \cong R \Rightarrow I^{(tp^e)} = (I^{(p^e)})^{(t)} \cong R$

$F^e_* R \cong R \oplus$ — $\quad$ apply $- \otimes I^{(t)}$

☆ $F^e_* R \otimes_R I^{(t)} \cong I^{(t)} \oplus$ — $\quad$ apply $(-)^{**}$ where $(-)^* = \mathrm{Hom}(-, R)$

LHS ☆: $\cong F^e_* I^{(tp^e)} \cong F^e_* R$
RHS ☆: $\cong I^{(t)} \oplus$ —

$F^e_* R \cong I^{(t)} \oplus$ —

$\Rightarrow I^{(t)}$ is a C-M R-module

$\Rightarrow \dfrac{I^{(t)}}{f I^{(t)}}$ is a C-M $R/\langle f \rangle$-module

$$\frac{I^{(t)}}{f I^{(t)}} \cong \frac{(I^{(t)}, f)}{(f)} \overset{\text{b/c } (S_2) \text{ as } R/\langle f \rangle\text{-mod}}{=} \left(\frac{I, f}{(f)}\right)^{(t)}.$$