CHAMPS 2021

Minimal DG Algebra Resolutions of Certain Stanley-Reisner Rings

GOALS

① Convince you of the value of $\hat{\Delta}$

② Showcase a minimal free resolution

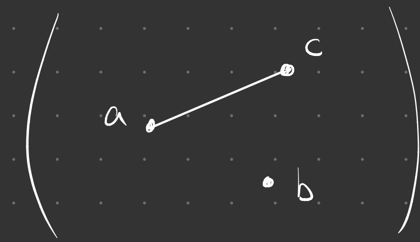
③ Showcase the DG algebra structure

$\boxed{\text{Goal 1 : } \hat{\Delta}}$

Definition: A <u>simplicial complex</u> Δ on a vertex set V is a collection of subsets of V which is closed under taking subsets. The elements of Δ are called the <u>faces</u> of the complex and the maximal faces are the <u>facets</u>.

Example: $V = \{a, b, c\}$

$$\Delta = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,c\}\} = \{\emptyset, a, b, c, ac\} = \langle ac, b \rangle =$$



Definition: A simplicial complex is <u>pure</u> if all its facets are the same size. If Δ is a simplicial complex on $V = \{a_1, \dots, a_n\}$, then let $\hat{V} = V \cup \{\alpha_1, \dots, \alpha_n\}$ be a vertex set and for each face $F \in \Delta$ define

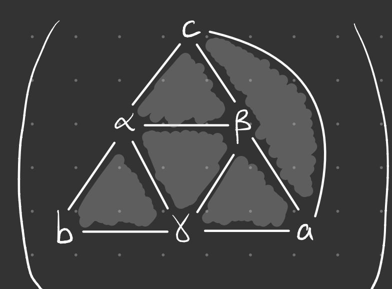
$$\hat{F} = F \cup \{\alpha_i \in \hat{V} \mid a_i \notin F\}.$$

Let $\hat{\Delta}$ be the simplicial complex on $\hat{V}$ generated by all such $\hat{F}$ and call it the <u>purification</u> of Δ.

Example: $\hat{V} = V \cup \{\alpha, \beta, \gamma\}$, $\Delta = \{\phi, a, b, c, ac\}$

F	$F^c \subset V$	convert to Greeks	combine
ϕ	abc	$\alpha\beta\gamma$	$\alpha\beta\gamma$
a	bc	$\beta\gamma$	$a\beta\gamma$
ac	b	β	$a\beta c$

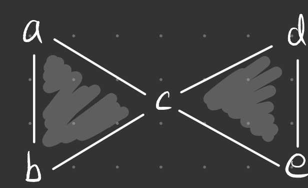
$\therefore \quad \hat{\Delta} = \langle \alpha\beta\gamma,\ a\beta\gamma,\ \alpha b\gamma,\ \alpha\beta c,\ a\beta c \rangle =$



Definition: a pure simplicial complex is <u>shellable</u> if one can list its facets $F_1, F_2, \dots, F_p$ such that $\dim(\langle F_i \rangle \cap \langle F_1, \dots, F_{i-1} \rangle) = \dim(F_i) - 1$ for every $i = 2, \dots, p$.

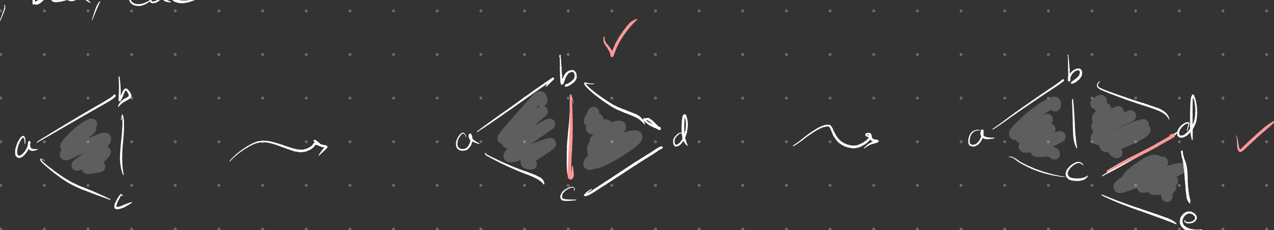
Example:

shellable



not shellable

shelling: abc, bcd, cde

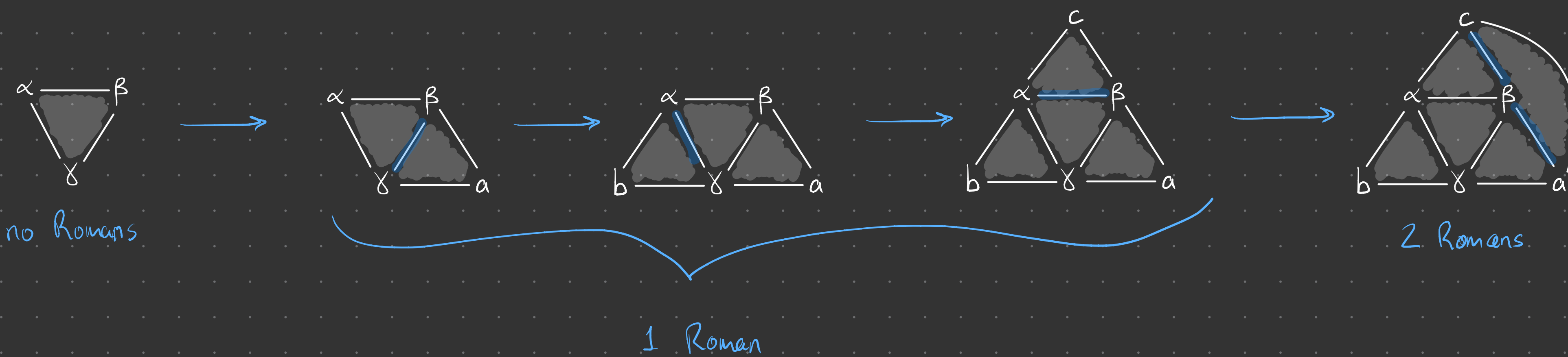


not a shelling: abc, cde, bcd

Goal 1 : $\hat{\Delta}$

Theorem (Morra) $\hat{\Delta}$ is shellable. Specifically, if the facets of $\hat{\Delta}$ are listed in order of increasing number of Romans, then that list is a shelling.

Example: $\alpha\beta\gamma$, $a\beta\gamma$, $\alpha b\gamma$, $\alpha\beta c$, $a\beta c$ is a shelling



Remark: shellable $\Longrightarrow$ C-M [Bruns & Herzog, Theorem 5.1.13]

Goal 1 : $\hat{\Delta}$

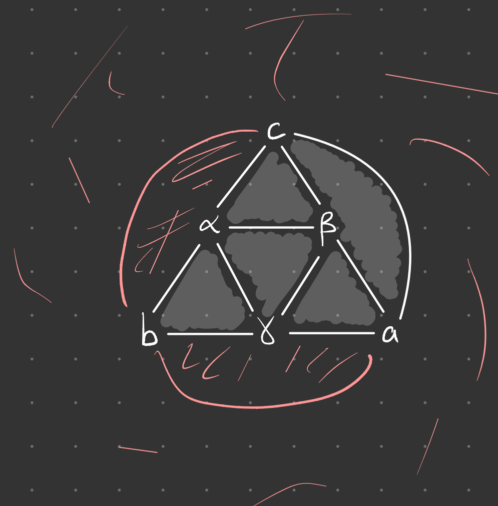
Definition: Let Δ and $\hat{\Delta}$ be as above, identify a variable with each vertex in $\hat{V}$, and define the polynomial ring $R = k[a_1, \dots, a_n, \alpha_1, \dots, \alpha_n]$. The Stanley-Reisner ideal of $\hat{\Delta}$ is generated by the non-faces of $\hat{\Delta}$, ie,

$$J_{\hat{\Delta}} = \langle m_F \mid F \subset \hat{V},\ F \notin \hat{\Delta} \rangle \trianglelefteq R$$

where $m_F = \prod_{v \in F} v$. The Stanley-Reisner ring of $\hat{\Delta}$ is $S = R/J_{\hat{\Delta}}$.

Example: If $\hat{\Delta} = \langle \alpha\beta\gamma, a\beta\gamma, \alpha b\gamma, \alpha\beta c, a\beta c \rangle$, then

$$\begin{aligned} J_{\hat{\Delta}} &= \langle a\alpha, b\beta, c\gamma, ab, bc, ab\gamma, \alpha bc, abc, \dots \rangle \\ &= \langle a\alpha, b\beta, c\gamma, ab, bc \rangle \\ &= \langle a, b, c \rangle \cap \langle \alpha, b, c \rangle \cap \langle a, \beta, c \rangle \cap \langle a, b, \gamma \rangle \cap \langle \alpha, b, \gamma \rangle \end{aligned}$$



Remark: $J_{\hat{\Delta}}$ is unmixed.

$\boxed{\text{Goal } 1 : \hat{\Delta}}$

Theorem (Morra): The minimal generators of the canonical module ω_S of S are in bijection with the facets of Δ.

Example: $\Delta = \langle b, ac \rangle \qquad \hat{\Delta} = \langle \alpha\beta\gamma, a\beta\gamma, \alpha b\gamma, \alpha\beta c, a\beta c \rangle$

$$\begin{tikzcd}
\{b\} \arrow[r, "\text{complement}"] & \{a, c\} \arrow[r, "\text{change to Greek}"] & \{\alpha, \gamma\} \arrow[dr] & \\
& & & \omega_S = \{\alpha\gamma, \beta\} \subseteq S \\
\{a, c\} \arrow[r] & \{b\} \arrow[r] & \{\beta\} \arrow[ur] &
\end{tikzcd}$$

Fact: If we localize S, e.g., $S_{\mathcal{X}}$ where $\mathcal{X} \subseteq S$ is generated by the variables, then the Cohen-Macaulay type of $S_{\mathcal{X}}$ is equal to the number of facets in Δ.

Corollary: In the special case when $J_{\hat{\Delta}}$ is the edge ideal of a K_1-corona ΣG of a simple graph G, the Cohen-Macaulay type of $S_{\mathcal{X}}$ is equal to the number of maximal cliques in G^c.

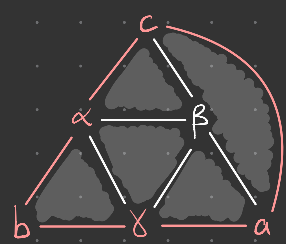
Goal 1 : $\hat{\Delta}$

Fact: Excluding the case when Δ is a simplex, $\hat{\Delta}$ is topologically equivalent to a ball, so its boundary is a sphere. [D'ali, Fløystad, Nematbaksh (2019)]

Example: Let Σ denote the boundary of $\hat{\Delta}$.

$$\hat{\Delta} = \langle \alpha\beta\gamma,\ a\beta\gamma,\ \alpha b\gamma,\ \alpha\beta c,\ a\beta c \rangle$$

$$\Sigma = \langle ac,\ a\gamma,\ b\gamma,\ \alpha b,\ \alpha c \rangle$$



$$\hat{\Delta} \setminus \Sigma = \{ \underbrace{\beta,}_{\mathcal{B}_3}\ \underbrace{\alpha\beta,\ \alpha\gamma,\ \beta\gamma,\ a\beta,\ \beta c,}_{\mathcal{B}_2}\ \underbrace{\alpha\beta\gamma,\ a\beta\gamma,\ \alpha b\gamma,\ \alpha\beta c,\ a\beta c}_{\mathcal{B}_1} \}$$

Goal 2: Resolution

Example: Let $\mathcal{L}$ denote the following resolution:

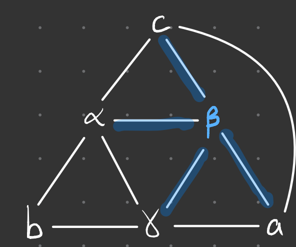
$$\underset{0}{S} \longleftarrow \underset{1}{S^{(\mathcal{B}_1)}} \longleftarrow \underset{2}{S^{(\mathcal{B}_2)}} \longleftarrow \underset{3}{S^{(\mathcal{B}_3)}} \longleftarrow 0.$$

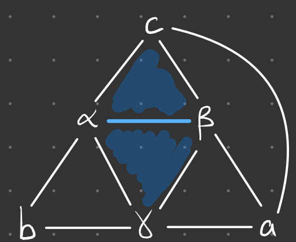
$[\alpha\beta\gamma]$ $[a\beta\gamma]$ $[\alpha b\gamma]$ $[\alpha\beta c]$ $[a\beta c]$

$[\alpha\beta]$ $[\alpha\gamma]$ $[\beta\gamma]$ $[a\beta]$ $[\beta c]$

$[\beta]$

$$\partial[\beta] = -\alpha[\alpha\beta] - a[a\beta] + \gamma[\beta\gamma] + c[\beta c]$$





$$\partial[\alpha\beta] = \gamma[\alpha\beta\gamma] + c[\alpha\beta c]$$

$$\partial[\alpha\beta\gamma] = abc \qquad \partial[a\beta\gamma] = -\alpha bc \qquad \partial[a\beta c] = \alpha b\gamma$$

Goal 2: Resolution

$\mathcal{L}$ minimally resolves S/I, where $I = (J_{\hat{\Delta}})^A$.

$$
S_0 \longleftarrow S_1^{(\mathcal{B}_1)} \longleftarrow S_2^{(\mathcal{B}_2)} \longleftarrow S_3^{(\mathcal{B}_3)} \longleftarrow 0
$$

S_1	S_2	S_3
$[\alpha\beta\gamma]$ $[a\beta\gamma]$ $[\alpha b\gamma]$	$[\alpha\beta]$ $[\alpha\gamma]$ $[\beta\gamma]$	$[\beta]$
$[\alpha\beta c]$ $[a\beta c]$	$[a\beta]$ $[\beta c]$	

$$
J_{\hat{\Delta}} = \langle a,b,c\rangle \cap \langle \alpha, b, c\rangle \cap \langle a, \beta, c\rangle \cap \langle a, b, \gamma\rangle \cap \langle \alpha, b, \gamma\rangle
$$

$$
\Rightarrow \quad I = (J_{\hat{\Delta}})^A = \langle abc, \alpha bc, a\beta c, ab\gamma, \alpha b\gamma\rangle
$$

Fact: By [DFN, Theorem 3.11], this is always a minimal resolution.

Definition: A commutative <u>differential graded S-algebra</u> is an S-complex

$$A = \left(0 \leftarrow A_0 \leftarrow A_1 \leftarrow A_2 \leftarrow \cdots\right)$$

equipped with a binary operator "$\cdot$" such that $|a\cdot b| = |a| + |b|$ and it

(i) is S-bilinear,

(ii) is unital, i.e., $\exists\, 1 \in A_0$ s.t. $1\cdot a = a = a \cdot 1$, $\forall a \in A$,

(iii) is associative,

(iv) is graded commutative, i.e.,

- $\forall a, b \in A \setminus \{0\}$ one has $b \cdot a = (-1)^{|a|\cdot|b|}\, a \cdot b$, and
- $|a|$ odd $\Longrightarrow$ $a \cdot a = 0$

(v) satisfies the Leibniz rule, i.e., $\forall a, b \in A\setminus\{0\}$ one has

$$\partial(a\cdot b) = \partial(a)\cdot b + (-1)^{|a|}\, a \cdot \partial(b).$$

Example: Koszul & Taylor.

Goal 3: DG algebra

Example:

$$S_0 \leftarrow S^{(\mathcal{B}_1)}_1 \leftarrow S^{(\mathcal{B}_2)}_2 \leftarrow S^{(\mathcal{B}_3)}_3 \leftarrow 0$$

$[\alpha\beta\gamma]$ $[a\beta\gamma]$ $[\alpha b\gamma]$ $[\alpha\beta c]$ $[a\beta c]$

$[\alpha\beta]$ $[\alpha\gamma]$ $[\beta\gamma]$ $[a\beta]$ $[\beta c]$

$[\beta]$

$$[\alpha\beta\gamma]\cdot[a\beta\gamma] = ?$$

$$\big|[\alpha\beta\gamma]\cdot[a\beta\gamma]\big| = \big|[\alpha\beta\gamma]\big| + \big|[a\beta\gamma]\big| = 2$$

$$\therefore\ [\alpha\beta\gamma][a\beta\gamma] = s_1[\alpha\beta] + s_2[\alpha\gamma] + s_3[\beta\gamma] + s_4[a\beta] + s_5[\beta c]$$

$$\therefore\ \partial\big([\alpha\beta\gamma][a\beta\gamma]\big) = s_1\cdot\partial([\alpha\beta]) + s_2\cdot\partial([\alpha\gamma]) + s_3\cdot\partial([\beta\gamma]) + s_4\cdot\partial([a\beta]) + s_5\cdot\partial([\beta c]) \qquad (1)$$

Leibniz:

$$\partial\big([\alpha\beta\gamma][a\beta\gamma]\big) = \partial([\alpha\beta\gamma])\cdot[a\beta\gamma] + (-1)^1[\alpha\beta\gamma]\cdot\partial([a\beta\gamma]) \qquad (2)$$

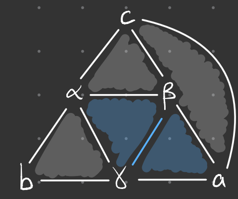
$$= abc[a\beta\gamma] - \alpha bc[\alpha\beta\gamma]$$

Much bookkeeping later:

$$[\alpha\beta\gamma][a\beta\gamma] = bc[\beta\gamma].$$

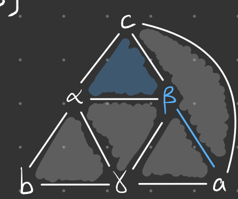
Goal 3: DG algebra

Example: $[\alpha\beta\gamma][a\beta\gamma] = bc[\beta\gamma]$ $\quad \alpha\beta\gamma \cap a\beta\gamma = \beta\gamma \in \hat{\Delta}\setminus\Sigma$ and $|[\beta\gamma]| = 2$ ✓



graded commutivity $\Longrightarrow$ $[a\beta\gamma][\alpha\beta\gamma] = -(1)^{1\cdot 1} bc[\beta\gamma] = -bc[\beta\gamma]$

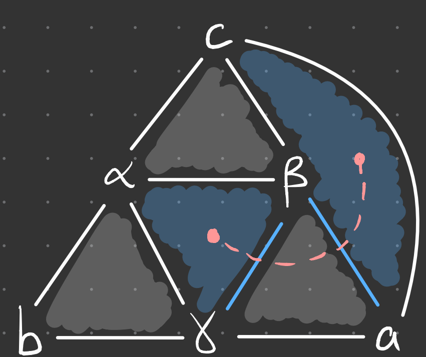
Example: $\alpha\beta c \cap a\beta = \beta$ and $|[\alpha\beta c]| + |[a\beta]| = 1+2 = |[\beta]|$ $\therefore$ $[\alpha\beta c][a\beta] = b\gamma[\beta]$



graded commutivity $\Longrightarrow$ $[a\beta][\alpha\beta c] = (-1)^{1\cdot 2} b\gamma[\beta] = b\gamma[\beta]$

Example: $[\alpha\beta\gamma][a\beta c] = -b\gamma[\beta\gamma] + ab[a\beta]$

$\alpha\beta\gamma \cap a\beta c = \beta \in \hat{\Delta}\setminus\Sigma$ ✓ $\qquad |[\beta]| = 3 \neq |[\alpha\beta\gamma]| + |[a\beta c]|$ ✗



$\alpha\beta\gamma,\ a\beta\gamma,\ a\beta c$

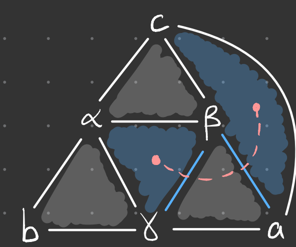
Discussion.

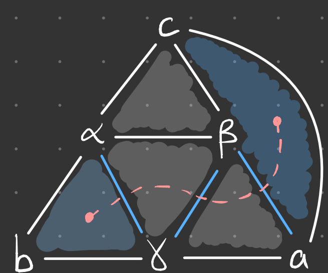
① When $F \cap F'$ is of appropriate codimension and is an element of $\hat{\Delta} \setminus \Sigma$, the product $[F][F']$ is simple. $\pm (F^c \cap F'^c)\,[F' \cap F]$

② When $F \cap F'$ has the wrong codimension, the product $[F][F']$ is a linear combination of simple products that arise from a "path from F to F'." Such products we call complex.

$$[\alpha\beta\gamma][a\beta c] = (-1)^3 \left(\frac{\gamma}{c} [\alpha\beta\gamma][a\beta\gamma] + \frac{a}{\alpha} [a\beta\gamma][a\beta c] \right)$$

$$= -b\gamma[\beta\gamma] + ab[a\beta]$$





$$[\alpha b\gamma][a\beta c] = (-1)^4 \left(\frac{\alpha\gamma}{ac} [\alpha b\gamma][\alpha\beta\gamma] + \frac{\beta\gamma}{bc} [\alpha\beta\gamma][a\beta\gamma] + \frac{a\beta}{\alpha b} [a\beta\gamma][a\beta c] \right)$$

$$= -\alpha\gamma[\alpha\gamma] + \beta\gamma[\beta\gamma] - a\beta[a\beta]$$

Corollary: I is a Golod ideal.

Definition: For each $i = 1, \ldots, n$ define $\tau_i : \hat{V} \longrightarrow \hat{V}$ by

$$\tau_i(F) = (F \setminus \{a_i\}) \cup \{\alpha_i\}$$

if $a_i \in F$. Simlarly, we define $t_i : \hat{V} \longrightarrow \hat{V}$ by

$$t_i(F) = (F \setminus \{\alpha_i\}) \cup \{a_i\}$$

if $\alpha_i \in F$.

Example:

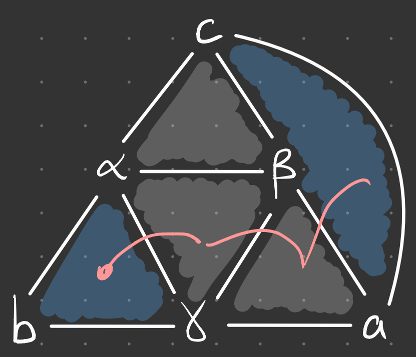
$$\alpha b \gamma \xmapsto{\tau_2} \alpha \beta \gamma \xmapsto{t_1} a \beta \gamma \xmapsto{t_3} a \beta c$$

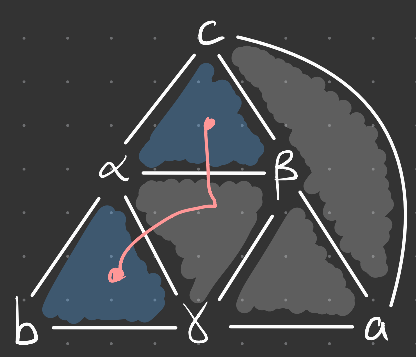
$$P(\alpha b \gamma, a\beta c) = \{\alpha b \gamma, \alpha \beta \gamma, a \beta \gamma, a \beta c\}$$

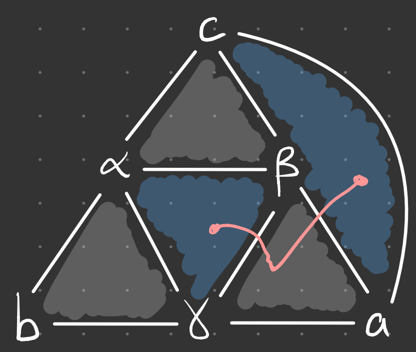
"path from $\alpha b \gamma$ to $a \beta c$"

Remark: When forming paths between facets, we apply these maps as-needed in this order:

$$\tau_n, \tau_{n-1}, \ldots, \tau_1, t_1, t_2, \ldots, t_n.$$







Discussion. Many products are zero under this structure, including some when the intersection is of appropriate codimension, e.g.,

$$[\alpha\beta c][a\beta] = b\gamma[\beta], \qquad \text{but} \qquad [\alpha\beta c][\beta\gamma] = 0,$$

even though we have

$$\left|[\alpha\beta c \cap a\beta]\right| = \left|[\beta]\right| = 3 \qquad \left|[\alpha\beta c \cap \beta\gamma]\right| = \left|[\beta]\right| = 3.$$

It turns out that for each face $F \in \hat{\Delta} \setminus \Sigma$ we can define a set $\varepsilon(F) \subset \mathcal{P}(\hat{V})$ such that under a few reasonable conditions we have

$$[F][F'] \neq 0 \qquad \text{iff} \qquad F' \in \hat{\Delta} \setminus \Sigma \text{ contains an element of } \varepsilon(F).$$

Example: $\varepsilon(\alpha\beta c) = \{\alpha\beta\gamma, a\beta, b\} \subset \mathcal{P}(\hat{V})$

$$\therefore [\alpha\beta c][a\beta] \neq 0 \qquad \text{and} \qquad [\alpha\beta c][\beta\gamma] = 0$$

Example: $\varepsilon(a\beta) = ?$ a bit tedious

Thank You.