CHAMP Seminar
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Branched covers and matrix factorizations.

Def. Let S be a regular local ring, $0 \neq f \in S$ a non-unit. A *matrix factorization of f* is a pair of square matrices (φ, ψ) with entries in S such that

$$\varphi\psi = f \cdot I_n = \psi\varphi.$$

Ex $f = x^3 + y^4 \in S = K[[x,y]]$

$$\underbrace{\begin{pmatrix} x & -y^2 \\ y^2 & x^2 \end{pmatrix}}_{\varphi} \cdot \underbrace{\begin{pmatrix} x^2 & y^2 \\ -y^2 & x \end{pmatrix}}_{\psi} = \begin{pmatrix} x^3+y^4 & 0 \\ 0 & x^3+y^4 \end{pmatrix}$$

So, (φ, ψ) is a MF of x^3+y^4 of size 2.

Let $R = S/(f)$ be the *hypersurface ring* defined by f.

MFs and R-modules

Category of MFs of f

Two observations. Let $(\varphi, \psi) \in MF(f)$

1) Since $\varphi\psi = \psi\varphi = f \cdot I_n$, both φ and ψ are injective

Have $0 \longrightarrow S^n \xrightarrow{\varphi} S^n \longrightarrow \operatorname{Cok}\varphi \longrightarrow 0$

2) $f(S^n) = \varphi\psi(S^n) \subseteq \operatorname{Im}\varphi$

$\Rightarrow$ $\operatorname{Cok}\varphi$ is naturally an R-module

Recall

- A finitely generated module M over a local ring A is maximal Cohen-Macaulay (MCM) if

$$\operatorname{depth} M = \dim A$$

- Auslander - Buchsbaum: If $\operatorname{pd}_A M < \infty$, then

$$\operatorname{depth} M = \operatorname{depth} A - \operatorname{pd}_A M$$

Given a MF of f (φ, ψ)

$$\operatorname{depth} \operatorname{Cok}\varphi = \dim S - \underbrace{\operatorname{pd}_S \operatorname{Cok}\varphi}_{1} = \dim R$$

So: $\operatorname{Cok}\varphi$ is an MCM R-module.

Conversely: An MCM R-module M has $\operatorname{pd}_S M = 1$

Have

$$\begin{tikzcd}
0 \arrow[r] & S^n \arrow[r, "\varphi"] \arrow[d, "f"] & S^n \arrow[r] \arrow[d, "f"] \arrow[dl, dashed, "\psi"'] & M \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & S^n \arrow[r, "\varphi"'] & S^n \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

$(\varphi, \psi) \in MF(f)$.

Thm (Eisenbud 80') The functor

$$\begin{aligned} MF(f) &\longrightarrow MCM\,R \\ (\varphi, \psi) &\longmapsto \operatorname{Cok}\varphi \end{aligned}$$

induces a bijection between reduced MFs and MCM R-mods with no free summands.

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φ, ψ have entries in max'l ideal of S.

Representation theory of hypersurface rings.

- A local ring A has finite CM type if, up to isomorphism, there are only finitely many indecomposable MCM R-modules.
- For a hypersurface, this is equivalent to saying there are only finitely many indecomposable MFs of f.

Ex| The hypersurface ring $R = K[\![x,y]\!]/(x^2)$ is not of finite CM type.

Here are infinitely many non-iso indecomp MFs of x^2

[BGS]: $\left(\begin{pmatrix} x & y^n \\ 0 & -x \end{pmatrix}, \begin{pmatrix} x & y^n \\ 0 & -x \end{pmatrix} \right) \quad n \geq 1$.

Double Branched Cover. $(S, \mathfrak{n}, K)$ w/ char $K \neq 2$.

If $R = S/(f)$, then $R^{\#} = S[\![z]\!]/(f+z^2)$ is the double branched cover of R.

Thm (Knörrer)

R finite CM type $\iff$ $R^{\#}$ finite CM type

$\Updownarrow$ (Eisenbud) $\qquad\qquad$ $\Updownarrow$ (Eisenbud)

$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

f finite MF type $\iff$ $f + z^2$ finite MF type

Cor: Simple hypersurface singularities of any dimension (ADE) have finite CM type.

Key Ingredient. $(S, \mathfrak{n}, K)$ complete RLR, char $K \neq 2$, $K = \bar{K}$

- $R = S/(f)$
- $R^\# = S[\![z]\!]/(f+z^2)$
- $\sigma : R^\# \longrightarrow R^\# \in \mathrm{Aut}(R^\#)$

  $\sigma(s) = s$

  $\sigma(z) = -z$

Note: $\sigma^2 = 1_{R^\#}$

Form the skew group algebra $R^\#[\sigma]$

- formal sums $a + b\sigma$ $a, b \in R^\#$
- multiplication given by: $a, b \in R^\#$

$$(a \cdot \sigma^i) \cdot (b \cdot \sigma^j) = a\sigma^i(b) \cdot \sigma^{i+j}$$

Thm (Knörrer) $\mathrm{MF}(f) \simeq \mathrm{MCM}(R^\#[\sigma])$

$$= \left\{ \begin{array}{l} R^\#[\sigma]\text{-mods which are} \\ \text{MCM over } R^\# \end{array} \right\}$$

$$R^\# \hookrightarrow R^\#[\sigma]$$

$r \longmapsto r \cdot 1_{R^\#}$

$$\begin{matrix} \text{Rep type} & \longleftrightarrow & \text{Rep type} & \longleftrightarrow & \text{CM type} \\ MF(f) & & MCM(R^\#[\sigma]) & & R^\# \end{matrix}$$

Motivating Question: What if we consider $f + z^d$ for $d > 2$? $*$

d-fold Matrix Factorizations

Def. Fix $d \geq 2$. (S, n, K) complete RLR, char $K \nmid d$, $K = \bar{K}$

$f \in S$ non-zero non-unit.

A matrix factorization of f with d factors is a tuple $(\varphi_1, \varphi_2, \ldots, \varphi_d)$ of $n \times n$ matrices with entries in S s.t.

$$\varphi_1 \varphi_2 \cdots \varphi_d = f \cdot I_n$$

Notice: $\varphi_i \varphi_{i+1} \cdots \varphi_d \varphi_1 \cdots \varphi_{i-1} = f \cdot I_n$ for all i

$MF^d(f)$ = category of d-fold MFs of f.

Trivial Example. $f \in S$, $d = 3$

$(f, 1, 1) \qquad (1, f, 1) \qquad (1, 1, f)$

1×1 MF of f w/ 3 factors.

Side note: • These are precisely the indecomposable projective (true for all $d \geq 2$) objects in $MF^3(f)$

- They are also injectives and $MF^3(f)$ is Frobenius.

Ex: $f = x^3 + y^4 \in S = K[\![x,y]\!]$. Assume $\omega \in K$ is a prim 3^{rd} root of 1.

$$\left(\begin{pmatrix} y^2 & 0 & x \\ x & y & 0 \\ 0 & x & y \end{pmatrix}, \begin{pmatrix} y & 0 & \omega x \\ \omega x & y & 0 \\ 0 & \omega x & y^2 \end{pmatrix}, \begin{pmatrix} y & 0 & \omega^2 x \\ \omega^2 x & y^2 & 0 \\ 0 & \omega^2 x & y \end{pmatrix} \right)$$

is a 3-fold MF of $x^3 + y^4$.

Thm (–) $f \in S$, $d \geq 2$, $\omega \in S$ is a primitive d^{th} root of 1.

- $R = S/(f)$
- $R^\# = S[\![z]\!]/(f+z^d)$ ⟵ d-fold branched cover of R
- $\exists$ $\sigma : R^\# \to R^\# \in Aut(R^\#)$
  $\sigma(s) = s$, $s \in S$ Notice $\sigma^d = 1_{R^\#}$
  $\sigma(z) = \omega z$

Form $R^\#[\sigma]$ as before. Then,

$$MF^d(f) \simeq MCM(R^\#[\sigma])$$

$$= \left\{ \begin{array}{l} R^\#[\sigma]\text{-modules which are} \\ \text{MCM over } R^\# \end{array} \right\}$$

Properties of $\Gamma = R^\#[\sigma]$.

1) $\operatorname{injdim}_{\Gamma}\Gamma = \operatorname{injdim}\Gamma_{\Gamma} = \dim S$. (Gorenstein-like)

2) Any f.g. Γ-module M has a projective resolution which is eventually 2-periodic.

$R^\#[\sigma] \sim$ non-commutative hypersurface

The idea behind the equivalence:

Let $N \in MCM(R^\#[\sigma]) \subseteq MCM\, R^\# \Rightarrow N$ is f.g. free over S.

Let $\varphi : N \to N$ be multiplication by z. Pick an S-basis for N and write φ as a matrix $\underbrace{\text{with entries in } S}_{\text{no } z \text{ entries}}$.

Then, $\varphi^d = \text{mult by } z^d = -f \cdot I_n$.

Get a MF of $f \approx (\underbrace{\varphi, \varphi, \dots, \varphi}_{d \text{ times}})$ with d factors $\left(\begin{matrix}\text{of size} \\ \operatorname{rank}_S N\end{matrix}\right)$

Notice this applies to any MCM $R^\#$-module.

$$\begin{tikzcd}
MCM(R^\#) \arrow[d, shift right] \arrow[rr, bend left, "\flat"] & & \\
MCM(R^\#[\sigma]) \arrow[u, shift right] \arrow[rr, "\sim"] & & MF^d(f) \arrow[llu, bend right, "\#"']
\end{tikzcd}$$

$\#$ and $\flat$ do not form an equivalence but:

(Leuschke, –) Let $N \in MCM\, R^\#$ and $X \in MF^d(f)$.

$$N^{\flat\#} \cong \bigoplus_{i=0}^{d-1} (\sigma^i)^* N \quad , \quad X^{\#\flat} \cong \bigoplus_{i=0}^{d-1} T^i X$$

where

- $(\sigma^i)^* N$ is the module obtained by restricting scalars along $\sigma^i : R^\# \to R^\#$
- $T^i(\varphi_1, \varphi_2, \ldots, \varphi_d) = (\varphi_i, \varphi_{i+1}, \ldots, \varphi_d, \varphi_1, \ldots, \varphi_{i-1})$

Say that f has finite d-MF type if there are, up to iso, finitely many indecomposable d-fold MFs of f.

Thm (Leuschke, –)

f has finite d-MF type iff $R^\# = S[\![z]\!]/(f+z^d)$ has finite CM type.

Corollary. $S = K[\![y, z_2, z_3, \ldots, z_r]\!]$, $K = \bar{K}$, char $K = 0$, and $d > 2$. Then f has finite d-MF type iff f and d are one of:

(A_1)	$y^2 + z_2^2 + \cdots + z_r^2$	any $d > 2$
(A_2)	$y^3 + z_2^2 + \cdots + z_r^2$	$d = 3, 4, 5$
(A_3)	$y^4 + z_2^2 + \cdots + z_r^2$	$d = 3$
(A_4)	$y^5 + z_2^2 + \cdots + z_r^2$	$d = 3$