THE "SIZE" OF AN IDEAL

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- a new asymptotically defined numerical invariants

$$\mathrm{ht}(I) \leq \mathrm{size}(I) \leq \mathrm{ara}(I)$$

↑ arithmetic rank

- one may use size to attack problems in set-theoretical intersections.
- one may use symbolic powers to calculate size. The consequence is yet to be further explored.

Outline: I. Quasilength

II. Size

III. Size = height

IV. Techincal difficulties in quasilength

I. Quasilength

DEF: R a ring, $I \subseteq R$ an ideal, M a R-module

M has finite I-quasilength if M has a finite filtration in which factors are R/I-cyclic modules

In this case, $\mathcal{L}_I(M)$ = the length of a shortest such filtrations

RMK: if I is a maximal ideal, $\mathcal{L}_I = \ell_I$ ↖ length

PROP: R a ring, I f.g. ideal $\subseteq R$, M, M_1, M_2, M_3 R-modules

(i) M has finite I-quasilength iff M is f.g. as an R-module & is killed by a power of I

$$\nu(M) \leq \mathcal{L}_I(M)$$

↑ the least # of generators

(ii) $0 \to M_1 \to M_2 \to M_3 \to 0$ SES

M_2 has finite I-quasilength iff both M_1 & M_3 do

$$\mathcal{L}_I(M_2) \leqslant \mathcal{L}_I(M_1) + \mathcal{L}_I(M_3)$$

$$\mathcal{L}_I(M_2) \geqslant \mathcal{L}_I(M_3) \qquad \cancel{\mathcal{L}_I(M_2) \geqslant \mathcal{L}_I(M_1)}$$

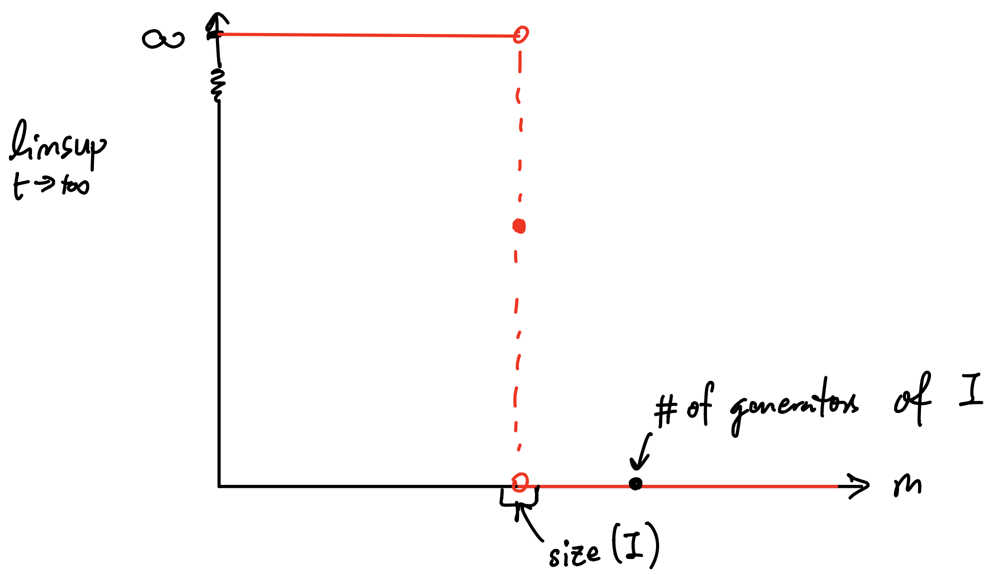
Caveat: the equality usually DOES NOT hold.

(iii) If S is an R-algebra, then $\mathcal{L}^S_{IS}(S \otimes_R M) \leqslant \mathcal{L}^R_I(M)$

(iv) If $I = (x_1, \cdots, x_n)$, then $\mathcal{L}_I\left(R/I^{t+1}\right) \leqslant \underbrace{\binom{n+t}{t}}_{\sim t^n}$

II. Size

DEF: $$\operatorname{size}_R(I) = \inf\left\{ m \;\middle|\; \limsup_{t \to +\infty} \frac{\mathcal{L}_I(R/I^t)}{t^m} < \infty \right\}$$



- $\operatorname{size}(I) \leqslant \nu(I)$
- $\operatorname{size}_S(IS) \leqslant \operatorname{size}_R(I)$

II.(1) Upper bounds

LEM: R a noetherian ring
$I, J \subseteq R$ ideals such that $\operatorname{rad}(I) = \operatorname{rad}(J)$

Then there $\exists$ $C_1, C_2 > 0$ s.t. for any module M

of finite I-quasilength

$$C_1 \mathcal{L}_I(M) \leq \mathcal{L}_J(M) \leq C_2 \cdot \mathcal{L}_I(M)$$

PROOF: Write $K = \operatorname{rad}(I)$

$K^n \subseteq I \subseteq K \rightsquigarrow R/K^{(n)} \twoheadrightarrow R/I \twoheadrightarrow R/K$

$\Uparrow$ has a finite K-quasilength

$\#$

PROP: Let $I, J, K \subseteq R$ ideals, Then

(i) If $\operatorname{rad}(J) = \operatorname{rad}(I)$, then

$$\operatorname{size}(I) = \inf\left\{ m \;\middle|\; \limsup_{t\to\infty} \frac{\mathcal{L}_J(R/I^t)}{t^m} < \infty \right\}$$

(ii) If $I \subseteq J$, $\wedge^{\operatorname{rad}(J)=\operatorname{rad}(I)}$ $\operatorname{size}(I) \geq \operatorname{size}(J)$ $\rightarrow$ $I^t \subseteq J^t \Rightarrow R/I^t \twoheadrightarrow R/J^t$

$$\limsup_{t\to\infty} \frac{\mathcal{L}_I(R/I^t)}{t^m} \geq \limsup_{t\to\infty} \frac{\mathcal{L}_I(R/J^t)}{t^m}$$

(iii) $\operatorname{size}(I^h) = \operatorname{size}(I)$

(iv) $I \subseteq J \subseteq K$, $\operatorname{size}(I) = \operatorname{size}(K) \Rightarrow \operatorname{size}(I) = \operatorname{size}(J) = \operatorname{size}(K)$

RMK: We can use any sequence of ideals $\{I_t\}$ s.t. $I^{c_1 t} \subseteq I_t \subseteq I^{c_2 t}$ to calculate the size of I.

$\Rightarrow$ If $P \in R$ is a prime ideal, P-adic top coincides with $P^{(n)}$-top, then we can calculate the size of P using $\{P^{(n)}\}_n$

THM: The notion of size is invariant up to radicals.

PROOF: $K = \operatorname{rad}(I)$ $\quad K^n \subseteq I \subseteq K$

$\operatorname{size}(K^n) \geq \operatorname{size}(I) \geq \operatorname{size}(K)$

$\operatorname{size}(K^n) = \operatorname{size}(K)$

$\Rightarrow \operatorname{size}(I) = \operatorname{size}(K)$

$$\text{size}(I) \leq \underbrace{\nu(\text{any ideal having the same radical of } I)}_{\text{ara}(I)}$$

II.(2) Lower bounds and nilpotents

PROP: R a noetherian ring, $I \subseteq R$ an ideal
P a minimal prime of I of height h
Then $\text{size}(I) \geq h$

PROOF: $R \to R_P$
IR_P is PR_P-primary
$\text{size}(I) \geq \text{size}(IR_P) = h$

the growth of the length of $R_P/(IR_P)^t$ which grows as a deg h poly in t

$\text{size}(I) \geq \text{ht}(I)$

$(\geq \text{super ht}(I)$ $\quad$ super ht = largest height of IS in any R-alg S $)$

LEM: f is a nilpotent element in R
$I \subseteq R$ an ideal.
$\bar{R} = R/fR \qquad \bar{I} = I\bar{R}$
Then $\text{size}_{\bar{R}}(\bar{I}) = \text{size}_R(I)$

THM: R a noetherian ring. For any $I \subseteq R$
$\text{size}_R(I) = \text{size}_{R_{red}}(IR_{red}) \qquad R_{red} = R/\text{nilrad}(R)$

PROP: $I \subseteq R$ a f.g. ideal. $\text{size}(I) = 0 \iff I$ is nilpotent.

III. Size = height

THM: R a local noetherian ring
$P \subseteq R$ a prime ideal such that $\dim R/P = 1$
There exists c s.t. $P^{(cn)} \subseteq P^n$ for all sufficiently large n
R/P is mod-fin over a regular local ring A
(e.g. R/P is complete)
Then $\mathrm{size}(P) = ht(P)$

PROOF: – $P^{(cn)} \subseteq P^n \subseteq P^{(n)}$

– a obvious filtration of $R/P^{(cn)}$

$$0 \subseteq P^{(cn-1)}/P^{(cn)} \subseteq P^{(cn-2)}/P^{(cn)} \subseteq \cdots \subseteq R/P^{(cn)}$$

each factor is torsion-free over R/P

– f.g. torsion-free A module are free A-module.

$\Downarrow$

growth at most $n^{ht(P)}$

$\Downarrow$

$\mathrm{size}(P) \leq ht(P) \Rightarrow \mathrm{size}(P) = ht(P)$.