Uniform bounds on symbolic powers in regular rings

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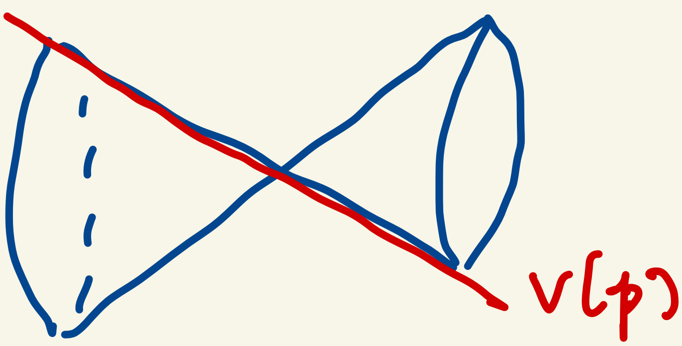
§1. Symbolic powers and the containment problem
§2. The proof in equal characteristic zero
§3. Ideas in the proof of Main Theorem

§1. Symbolic powers and the containment problem

Ex $R = k[x,y,z]/(xy - z^2)$



$\mathfrak{p} = (y, z) \subseteq R$ prime

NOT $\mathfrak{p}$-primary

$$\mathfrak{p}^2 = (y^2, yz, z^2) = (y^2, yz, xy)$$

$$= (y) \cap (x, yz, y^2)$$

[Krull '28]

You can enlarge p^2 so it *is* p-primary!

Def R = Noeth. ring
$\cup\!\!\mid$
I = ideal

n^{th} symbolic power of I

$$I^{(n)} := \bigcap_{p \in Ass(R/I)} (I^n R_p \cap R)$$

Note
- p prime $\Rightarrow$ $p^{(n)}$ is p-primary
- $I = I^{(1)}$
- $I^n \subseteq I^{(n)}$ ☆
- $I^{(n)} \supseteq I^{(n+1)} \supseteq \cdots$

Containment Problem

When do we have $I^{(m)} \subseteq I^n$?

Thm [Zariski '51]

R = Noeth. domain s.t. $\widehat{R_p}$ domain $\forall p$ prime

$\forall p, \forall n, \exists m$ s.t. $p^{(m)} \subseteq p^n$

In other words, the topologies defined by $\{p^n\}$ and $\{p^{(n)}\}$ are equivalent!

[Schenzel '85/'86]

Chacterized when $\{I^n\}$ and $\{I^{(n)}\}$ are equivalent.

Thm [Swanson '00] R = Noeth.

$\{I^n\}$ and $\{I^{(n)}\}$ are equivalent

$\iff \exists k$ s.t. $I^{(kn)} \subseteq I^n \ \forall n \geq 1$.

Q What is k?

Does it always depend on I?

Answers for regular rings:

Thm [Ein-Lazarsfeld-Smith '01;

Hochster-Huneke '02]

R = regular of equal char.

$\cup\!|$ $\qquad d = \dim(R)$

I = ideal

$\Rightarrow I^{(dn)} \subseteq I^n \ \forall n \geq 1$

Rem d can be replaced by:

$$\operatorname{bight}(I) := \max_{\mathfrak{p} \in \operatorname{Ass}(R/I)} \{\operatorname{ht}(\mathfrak{p})\}$$

$$\vee\!|$$

$$h := \text{largest analytic spread of } IR_{\mathfrak{p}} \text{ for } \mathfrak{p} \in \operatorname{Ass}(R/I).$$

Q [HH'02] Does this thm hold in mixed char.?

[Ma-Schwede '18] Yes, for radical ideals in excellent regular rings of mixed char.

[M-] Yes, in general

Main Thm [M-]

R = regular

$\cup$

I = ideal

new when
not nec'ly excellent
not nec'ly radical

$\Rightarrow I^{(hn)} \subseteq I^n \quad \forall n \geq 1$

Complete answer to [HH'02]!

Cor R = regular of finite Krull dim.

$\Rightarrow R$ satisfies the uniform symbolic topologies property of [Huneke-Katz-Validashti '15]

More generally,

Thm [M-] R = regular
$\cup|$
I = ideal

$s_1, \ldots, s_n \in \mathbb{Z}_{\geq 0}$
$s = \sum_i s_i$

ⓐ $I^{(s+nh)} \subseteq \prod_{i=1}^{n} I^{(s_i+1)}$

ⓑ If $R = (R, m)$ is local, then
$$I^{(s+nh+1)} \subseteq m \prod_{i=1}^{n} I^{(s_i+1)}$$

When R is of equal char.:

ⓐ HH'02; Johnson '14

ⓑ HH'07; Takagi-Yoshida '08; Johnson '14

ⓐ Application to Grifo's asymptotic version of Harbourne conj.

ⓑ Related to Eisenbud-Mazur conj.

How? · New version of perfectoid/big Cohen-Macaulay test ideals

· Recent developments in mixed char. CA + AG.

§2 Proof in equal char. zero

Main Thm [M-]

R = regular
$\cup\!|$
I = ideal

$\Rightarrow I^{(hn)} \subseteq I^n \quad \forall n \geq 1$

Pf when $R \supseteq \mathbb{Q}$ [ELS'01; M-]

· Reduce to complete local case

· Use multiplier ideals

$\mathcal{J}(R, I^t), \ t \in \mathbb{Q}_{\geq 0}$

"allow taking roots of ideals"

$$
\begin{aligned}
I^{(hn)} &\underset{①}{\subseteq} \mathcal{J}\left(R, [I^{(hn)}]^{1}\right) && \text{"not too small"} \\
&\underset{②}{\subseteq} \mathcal{J}\left(R, (I^{(hn)})^{n/n}\right) && \text{"unambiguity of exponents"} \\
&\underset{③}{\subseteq} \left(\mathcal{J}\left(R, (I^{(hn)})^{1/n}\right)\right)^{n} && \text{"subadditivity"} \\
&\underset{④}{\subseteq} I^{n} && \text{"not too large"}
\end{aligned}
$$

∎

<u>Rem</u>
- Need my Kawamata–Viehweg vanishing thm to do ④ [M-]
- Does not use Artin approximation

To define/use $\mathcal{J}(R, I^{t})$ need:
- resolution of sing's
- vanishing thm's

} unknown/false in pos./mixed char.

Idea [Hara '05; Ma-Schwede '18; M-]

Define test ideals $\tau(R, -)$ to replace

$\mathcal{J}(R, -)$

pos. char: Frobenius

mixed char: perfectoid geometry

[MS'18] need I radical in (4) to compute

$\tau(R, (I^{(hn)})^{1/n})$

(assumption on excellence is needed to make sure I stays radical when completing)

Need something new!

What will we need for 4?

4a) Localization (in equal char. 0)

$$\mathcal{J}(R, I^t) \cdot R_p = \mathcal{J}(R_p, IR_p^t)$$

4b) Skoda's Thm If $J \subseteq I$ is gen. by h elt's, and if $\overline{J} = \overline{I}$, then

$$\mathcal{J}(R, I^h) = J \cdot \mathcal{J}(R, I^{h-1})$$

Existing versions of test ideals in mixed char.

$(R, \mathfrak{m})$ regular cln

$\cup$|

$(f_1, \ldots, f_n) = I$

	[MS'18] $\tau(R, [\underline{f}]^t)$	[MS'21; Pérez-RG'21; Sato-Takagi] $\tau_B(R, I^t)$	[Hacon-Lamarche-Schwede] $\tau_+(R, I^t)$
① not too small	✓	✓	✓
② unambiguity	✓	✓	✓
③ subadditivity	✓	??	??
4a localization	??	??	principal OK for $\tau_+(R, \Delta)$
4b Skoda's thm	??	??	✓

None of these are sufficient to prove Main Thm alone!

Idea [M–] Find a new version of test ideals that combines the advantages of all of these

$\rightsquigarrow \tau_B(R, [\underline{f}]^t)$ satisfying ① ~ ④

§3 Ideas in proof of Main Thm

We will prove all cases of Main Thm, although the proof of ④ does split up into cases.

Key pt Big Cohen–Macaulay algebras do!

Why Homological conj.'s (incl. Direct summand conj.)

Def [Hochster '75; Sharp '81]

$(R, \mathfrak{m})$ = Noeth. local, dim d

B = R-alg.

B is (balanced) <u>big Cohen-Macaulay</u> (BCM)

if every s.o.p. $x_1, \ldots, x_d$ on R becomes

a regular sequence on B.

<u>Thm</u> [HH'92; Dietz-R.G. '19; André '18/'20;

Shimomoto '18]

Every Noeth. local domain (R, m) has a

BCM R^+-alg.

$R^+ :=$ int. closure of R in $\overline{\mathrm{Frac}(R)}$

<u>Thm</u> [HH'92; Bhatt]

$\widehat{R^+} =$ p-adic completion of R^+

is a BCM R^+-alg for Noeth. cl

domains of residue char. $p > 0$.

Def [M-] (nicer elt's from [MS'18 + '21])

$(R, \mathfrak{m})$ = regular clr, B = BCM R^+-alg.

Ψ

$f_1, \ldots, f_n$, $t \in \mathbb{Q}_{\geq 0}$

$\tau_B(R, [\underline{f}]^t)$

$$:= \operatorname{Ann}_{\omega_R} \left\{ \eta \in H^{d}_{\mathfrak{m}}(R) \;\middle|\; \begin{array}{l} \forall m \in \mathbb{Z}_{>0}, \\ g\eta = 0 \text{ in } H^d_{\mathfrak{m}}(B) \\ \forall g = \prod_{k=1}^{a} f_{j_k}^{1/m} \\ \text{where } a \geq mt \end{array} \right\}$$

$d = \dim(R)$

Compared to [MS'18]:

1. No almost math.
2. $m \in \mathbb{Z}_{>0}$ (not just p-powers)
3. no perturbations
4. works in all char.

Also: <u>Key comparison</u> [M-]

$B = \widehat{R}^+$ in rel. char. $p > 0$

$$\tau_{\widehat{R}^+}(R, [\underline{f}]^t)$$

$$\subseteq \sum_{m=1}^{\infty} \sum_{g \in I^m} \tau_t\left(R, \frac{t}{m} \operatorname{div}_R(g)\right)$$

$$\subseteq \tau_t(R, I^t)$$

[HLS]

☆ We can try to mimic strategy in char. 0 by switching to τ_t in the correct place!

Thm [M-] $\exists B$ s.t. $\boxed{\tilde{\tau}_B(R, [\underline{f}]^{hn}) \subseteq I}$

$(\underline{f}) = I^{(hn)}$

Pf If $R \supseteq \mathbb{Q}$,

$$\tilde{\tau}_B(R, [\underline{f}]^t) \subseteq \mathcal{J}(R, \underline{f}^{(hn)})^t)$$

So equal char. zero proof applies.

In res. char. $p > 0$:

STS: after localizing at every

$$p \in \mathrm{Ass}(R/I).$$

Instead we localize at $x \in R - p$ s.t.

- $\exists J \subseteq R$ gen. by h elts s.t.

$$\overline{J R_x} = \overline{I R_x}$$

- $I^{(hn)} R_x = I^{hn} R_x$

$$\tau_{\hat{R}^t}\left(R, [\underline{f}]^{1/n}\right) \cdot R_x$$

$$\underset{[M-]}{\subseteq} \sum_{m=1}^{\infty} \sum_{g \in [I^{hn}]^m} \tau_t\left(R, \frac{1}{nm} \operatorname{div}_R(g)\right) R_x$$

$$\subseteq \sum_{m=1}^{\infty} \sum_{g \in I^m} \tau_t\left(R, \frac{h}{m} \operatorname{div}_R(g)\right) R_x$$

$$\subseteq \sum_{m=1}^{\infty} \sum_{g \in \overline{J}^n} \tau_t\left(R, \frac{h}{m} \operatorname{div}_R(g)\right) R_x$$

$$\underset{[HLS]}{\subseteq} \tau_t\left(R, \overline{J}^{h}\right) \cdot R_x$$

$$= J \cdot \tau_t\left(R, \overline{J}^{h-1}\right) \cdot R_x$$

$$\subseteq I \cdot R_x \quad \blacksquare$$