Eloísa Grifo

William and Carolyn Polk Fellow

Put a ring (structure) on it

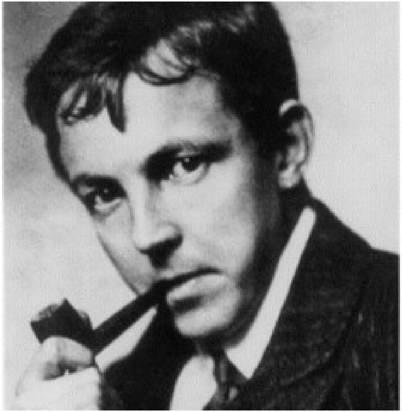
Jefferson Scholars Foundation

Commutative Algebra

What do mathematicians do?

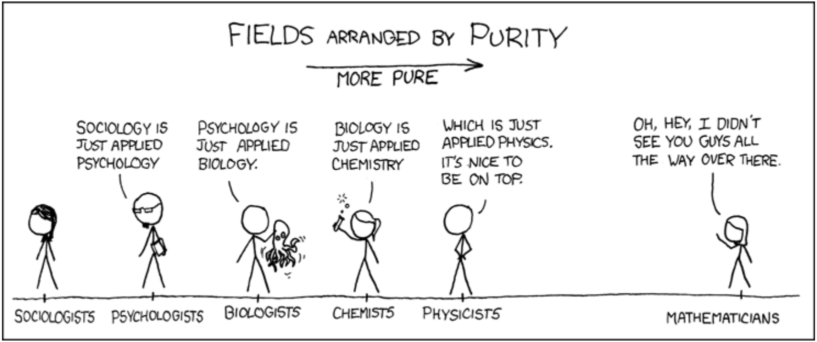

Δ=(-1)²-4·1·(-6)=1+24=25
√Δ = √25 = 5
(a+b)³=(a-b)(a²+2ab+b²)
c²=a²+b²
x=y²
A=<-2,8
B=(4,∞)
p·g·V

r2
3
09
7
23
21



G. H. Hardy (1877 – 1947)

"I have never done anything 'useful'. No discovery of mine has made, or is likely to make, directly or indirectly, for good or ill, the least difference to the amenity of the world."


FIELDS ARRANGED BY PURITY
MORE PURE
SOCIOLOGY IS JUST APPLIED PSYCHOLOGY
PSYCHOLOGY IS JUST APPLIED BIOLOGY.
BIOLOGY IS JUST APPLIED CHEMISTRY
WHICH IS JUST APPLIED PHYSICS. IT'S NICE TO BE ON TOP.
OH, HEY, I DIDN'T SEE YOU GUYS ALL THE WAY OVER THERE.
SOCIOLOGISTS
PSYCHOLOGISTS
BIOLOGISTS
CHEMISTS
PHYSICISTS
MATHEMATICIANS

Polynomials

Polynomials

Expressions like $x^2 - 1$, $5x^7 - 73x^3 + 42$, $xy + z^3$, etc.

We can sum polynomials

$$\left(x^3 + 2x + 1\right) + \left(5x^4 + 3x + 5\right) = 5x^4 + x^3 + 5x + 6.$$

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Polynomials

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We can multiply polynomials

$(2x+1)\left(5x^4+3x\right) = 10x^5 + 5x^4 + 6x^2 + 3x.$

We can sum integers too

When we sum two integers, the result is another integer.

Example

$123456789 + 987654321 = 1111111110.$

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Polynomials

The sum and multiplication of polynomials have nice properties that we have seen before.

Déjà vu

So maybe these are part of a larger construction!

What is a ring?

What is a ring?

Ring

A ring is a set R with two operations, $+$ and $\times$, verifying some nice properties.

Example

The main example to keep in mind is the integers with the usual sum and multiplication.

Properties of $+$ and $\times$

- $0 + a = a + 0 = a$.
- $1a = a1 = a$.
- $a + (-a) = 0$
- Commutativity: $a + b = b + a \qquad and \qquad ab = ba$
- Associativity:
  $$(a + b) + c = a + (b + c) \qquad \text{and} \qquad (ab)c = a(bc)$$
- Distributivity: $a(b + c) = ab + ac$

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Other examples of rings

- Polynomials.
- The rational numbers.
- The real numbers.
- The complex numbers.
- Functions.

We represent the addition and multiplication in tables:

$+$	0	1	2	3	4	5	$\cdots$
0	0	1	2	3	4	5	$\cdots$
1	1	2	3	4	5	6	$\cdots$
2	2	3	4	5	6	7	$\cdots$
3	3	4	5	6	7	8	$\cdots$
4	4	5	6	7	8	9	$\cdots$
5	5	6	7	8	9	10	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

$\times$	0	1	2	3	4	5	$\cdots$
0	0	0	0	0	0	0	$\cdots$
1	0	1	2	3	4	5	$\cdots$
2	0	2	4	6	8	10	$\cdots$
3	0	3	6	9	12	15	$\cdots$
4	0	4	8	12	16	20	$\cdots$
5	0	5	10	15	20	25	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

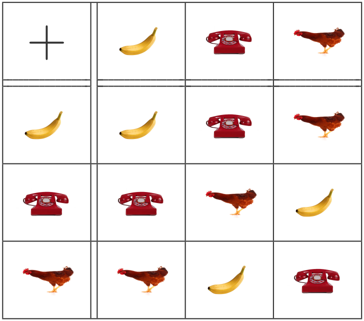
Maybe we can build a ring of our own

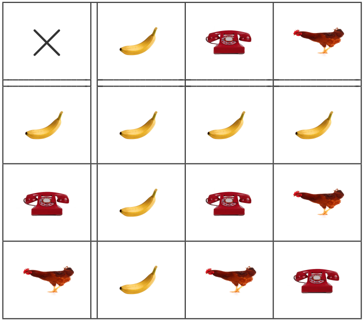
Banana

Telephone

Chicken





Building new rings

Quotients

Given a ring, we can build a new ring by gluing some elements together.

Multiples of 3

From the ring of integers $\mathbb{Z}$, we can:

- Glue all the multiples of 3 together.
- Glue all the numbers that when divided by 3, leave a remainder of 1, so the look like a multiple of 3 plus 1.
- Glue all the numbers that when divided by 3, leave a remainder of 2, so the look like a multiple of 3 plus 2.

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- Glue all the numbers that when divided by 3, leave a remainder of 2, so the look like a multiple of 3 plus 2.

Multiples of 3

To form the ring $\mathbb{Z}/3$, we use three elements:

- 0: represents all the multiples of 3.
- 1: represents all the numbers that leave remainder 1 when divided by 3.
- 2: represents all the numbers that leave remainder 2 when divided by 3.

Brave new ring

$\mathbb{Z}/3$ is a new ring with elements $0, 1$, and 2.

How do we sum and multiply them together?

$+$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

$\times$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

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1	1	2	0
2	2	0	1

$\times$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Replacing elements

$0 \longleftrightarrow$ 🍌

$+$	🍌	1	2
🍌	🍌	1	2
1	1	2	🍌
2	2	🍌	1

$\times$	🍌	1	2
🍌	🍌	🍌	🍌
1	🍌	1	2
2	🍌	2	1

Replacing elements

$0 \longleftrightarrow$  $1 \longleftrightarrow$ 

$+$			2
			2
		2	
2	2		

$\times$			2
			
			2
2		2	

Replacing elements

$0 \longleftrightarrow$  $1 \longleftrightarrow$  $2 \longleftrightarrow$ 

$+$			
			
			
			

$\times$			
			
			
			

Ideals

What is an ideal?

Ideals $\longleftrightarrow$ systems of equations

Ideal

An ideal is a subset of the ring that is nicely behaved. You can add elements of an ideal and stay inside the ideal, find negative elements, and even multiply an element of the ideal by any other element and stay inside the ideal.

Examples

The ideals of the ring of integers are the multiples of a fixed number.

Why are ideals nice?

We can use ideals to define quotients.

Examples

We used the ideal of multiples of 3 to define a quotient ring.

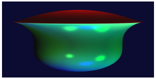
Why are ideals nice?

The ideals break the ring into nice layers. We can look at how many layers we have in a ring and measure the *dimension* of the ring.

Geometry and algebra

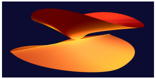
Surfaces

Can we tell geometric properties of these surfaces using algebra?



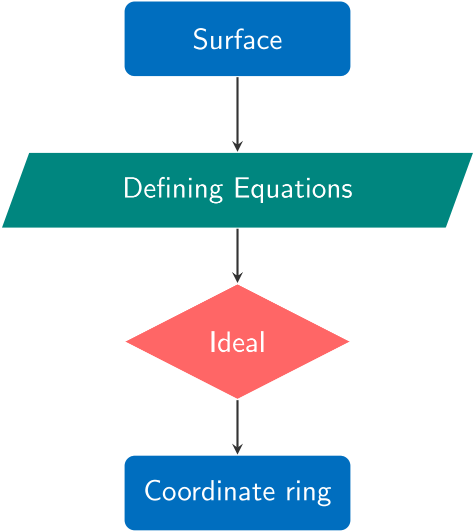
Twilight

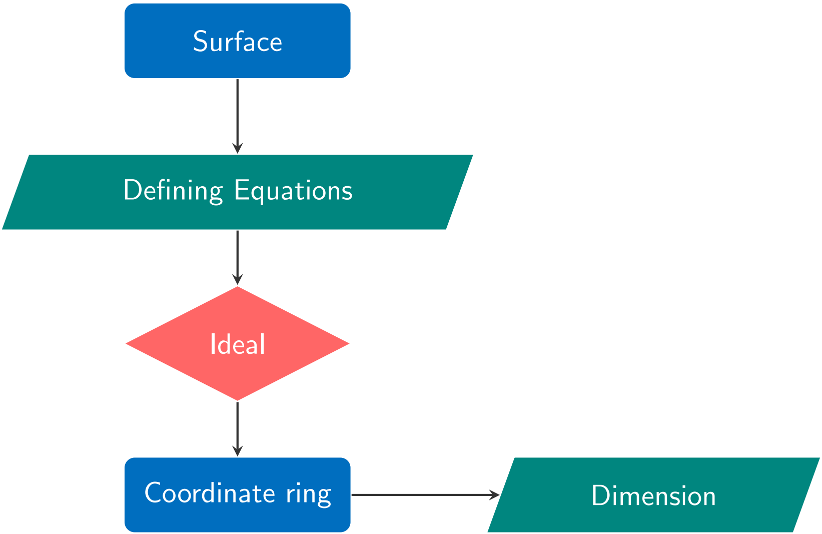
$$(z^2 - 2)^2 + (x^2 + y^2 - 3)^3 = 0$$

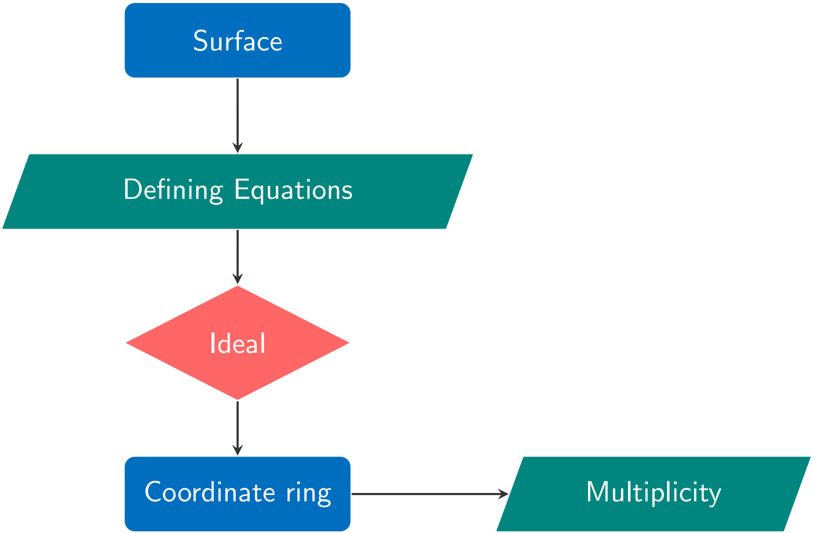


Calyx

$$z^4 = x^2 + y^2 z^3$$


Surface
Defining Equations
Ideal
Coordinate ring


Surface
Defining Equations
Ideal
Coordinate ring
Dimension


Surface
Defining Equations
Ideal
Coordinate ring
Multiplicity

Thank you!