Exercise 1.* Let $Q = k[\![x, y]\!]$. Find the minimal free resolution for $Q/(x^2, xy)$, with proof.

Exercise 2. Let $Q = k[\![x, y, z]\!]$ and $M = Q/(xy, yz, xz)$.

(a) Find $\operatorname{pdim}_Q(M)$ without writing the minimal free resolution for M.

(b) Find the minimal free resolution for M.

(c) Check your work with `Macaulay2`.

Exercise 3. For a dg algebra A, prove that the multiplication $[a] \cdot [b] := [a \cdot b]$ is well-defined and makes $\mathrm{H}(A)$ a graded algebra.

Exercise 4.* Write the Koszul complex on 3 elements f_1, f_2, f_3.

Exercise 5. Let I be a nonzero ideal in a noetherian local ring R. Show that if I is generated by a regular sequence, then I/I^2 is a free R-module.

Exercise 6. Let I be an ideal in a noetherian ring R generated by $\underline{x} = x_1, \dots, x_n$. Prove that the following hold.

(a) Each $\mathrm{H}_i(\mathrm{Kos}(\underline{x}))$ is finitely generated and $\mathrm{H}_i(\mathrm{Kos}(\underline{x})) = \begin{cases} 0 & i \notin [0, n] \\ R/I & i = 0 \\ \operatorname{ann}_R(I) & i = n\,. \end{cases}$

(b) Every Koszul homology module $\mathrm{H}_i(\mathrm{Kos}(\underline{x}))$ is killed by I.

Hint: One can directly show that multiplication by $a \in I$ is nullhomotopic. Alternatively, and perhaps more interestingly, one can use that each $\mathrm{H}_i(\mathrm{Kos}(\underline{x}))$ is a $\mathrm{H}_0(\mathrm{Kos}(\underline{x}))$-module.

Exercise 7.* For each of the following, describe the graded k-algebra structure on $\mathrm{H}(\mathrm{Kos}(x, y))$.

(a) $R = k[\![x, y]\!]/(x^2, xy)$.

(b) $R = k[\![x, y]\!]/(x^2, y^2)$.

Exercise 8.* Using your minimal free resolution F from Exercise 1, equip F with a dg Q-algebra structure. More generally, prove that if $I \subseteq \mathfrak{m}^2$ is minimally generated by f, g in a regular local ring Q, then the minimal free resolution of Q/I can be equipped with a dg Q-algebra structure.

Exercise 9.* Let $I \subseteq \mathfrak{m}^2$ be minimally generated by f, g in a regular local ring $(Q, \mathfrak{m}, k)$, and let F denote the minimal free resolution of Q/I equipped with a dg Q-algebra structure from Exercise 8. Describe the induced graded k-algebra structure on $F \otimes_Q k$; cf. Exercise 7.

Exercise 10. For a pair of R-complexes M, N, their Hom complex, denoted $\mathrm{Hom}_R(M, N)$, is the R-complex with

$$\mathrm{Hom}_R(M, N)_i = \prod_{j \in \mathbb{Z}} \mathrm{Hom}_R(M_j, N_{i+j}) \quad \text{and} \quad \partial^{\mathrm{Hom}}(\varphi) = \partial^N \circ \varphi - (-1)^{|\varphi|} \varphi \circ \partial^M\,.$$

(a) Check that $\mathrm{Hom}_R(M, N)$ is a complex.

(b) Describe $\mathrm{Z}_0(\mathrm{Hom}_R(M, N))$ and $\mathrm{H}_0(\mathrm{Hom}_R(M, N))$.

(c)* Set $\mathrm{End}_R(M) = \mathrm{Hom}_R(M, M)$. Prove that $\mathrm{End}_R(M)$ is a dg R-algebra.