Macaulay2 Remark: It is OK to use `Macaulay2` for difficult calculations. For example, for computing kernels of differentials.

Exercise 1.* Let x be an element of even degree in a graded commutative dg algebra with differential ∂. Show that

$$\partial(x^n) = n\partial(x)x^{n-1} \quad \text{for all } n \geqslant 1.$$

Exercise 2. Let A be a dg algebra and M a (left) dg A-module. Prove that for each i, the complex $\Sigma^i M$ has a (left) dg A-module structure where the A-action is given by

$$a \cdot \Sigma^i m := (-1)^{|a|i}\Sigma^i(am)\,.$$

Exercise 3.* Assume $\widehat{R} = Q/I$ is a minimal regular presentation. Let $Q[X]$ be a minimal model for R, and also let $\mathfrak{m}$ denote the maximal ideal of Q.

(a) Verify that $\partial(X_1) \subseteq \mathfrak{m}^2$ and $\partial(X_2) \subseteq \mathfrak{m}X_1$.

(b) Give an example showing where $\partial(X_{n+1}) \subseteq \mathfrak{m}X_n$ can fail to hold.

Exercise 4.* Let $Q = k[\![x, y]\!]$ and $Q[X]$ be the minimal model for $R = Q/(x^2, xy)$ over Q.

(a) Compute $Q[X_{\leqslant 3}]$ without using the `Macaulay2` package `DGAlgebras`.

(b) Check your work with the `Macaulay2` package `DGAlgebras`.

Exercise 5. Compute a minimal model for the local ring

$$R = k[\![x, y, z]\!]/(x^2 - yz, xz, y^2)$$

over $k[\![x, y, z]\!]$ up through degree 3 variables, without using the `Macaulay2` package `DGAlgebras`.

Exercise 6.* Let $R = k[t]/(t^2)$.

(a) Find the minimal model for the residue field of R when $k = \mathbb{Q}$, without using the `Macaulay2` package `DGAlgebras`.

(b) Does the answer change at all when $k = \mathbb{F}_2$?

Exercise 7.* Let F be a complex of free R-modules with $F_n = 0$ for all $n < 0$. Prove that if $\varphi\colon A \to B$ is a quasiisomorphism, then both of the induced maps $\varphi \otimes_R \mathrm{id}_F\colon A \otimes_R F \to B \otimes_R F$ and $\mathrm{Hom}_R(\mathrm{id}_F, \varphi)\colon \mathrm{Hom}_R(F, A) \to \mathrm{Hom}_R(F, B)$ are quasiisomorphisms.
Hint: The truncation subcomplexes $F_{\leqslant n}$ may be useful to consider, where $F_{\leqslant n}$ is the subcomplex of F with F_d in homological degrees $d \leqslant n$, and zeroes elsewhere.

Exercise 8. Let A be a graded commutative dg algebra and assume $\alpha\colon M \to N$ is a homotopy equivalence of dg A-modules. Prove that for any dg ideal J of A the induced map $\overline{\alpha}\colon M/JM \to N/JN$ is a homotopy equivalence of dg A/J-modules.

Exercise 9.* Let $R = \mathbb{F}_5[\![x, y, z]\!]/(x^2, y^2, z^2, xyz)$.

(a) Argue numerically that R is not a complete intersection.

(b) Use `Macaulay2` to compute $\mathrm{H}(K^R)$, as a graded $\mathbb{F}_5$-algebra, and use this information to explain why R is not a complete intersection ring.

Exercise 10. Let

$$R = \frac{\mathbb{F}_7[\![x, y, z, w]\!]}{(xy, z^2, zw, x^2z + w^3, y^2w + xw^2)}.$$

(a) Argue numerically that R is not a complete intersection.

(b) Use `Macaulay2` to compute $\mathrm{H}(K^R)$, as a graded $\mathbb{F}_7$-algebra, and use this information to explain why R is not a complete intersection ring.

Exercise 11.* Let p be prime and G an elementary abelian p-group (that is, $G = (\mathbb{Z}/p)^c$ for some c). Prove that the group algebra $\mathbb{F}_pG$ is an artinian complete intersection ring.

Exercise 12.!! (Wiebe's Theorem) Let R be a local ring and set $n = \mathrm{embdim}(R) - \mathrm{depth}(R)$. Then R is complete intersection if and only if $\mathrm{H}_1(K^R)^n \neq 0$.
Hint for ($\Longleftarrow$): First, reduce to the case that R has depth zero. Next, prove this assertion holds when R is artinian; in this case, factor a minimal regular presentation $Q \to S \to R$ with S artinian complete intersection and $\mathrm{H}_1(K^S)^n \to \mathrm{H}_1(K^R)^n$ is nonzero. The identification $\mathrm{H}_n(K^S) = \mathrm{soc}(S)$ and $\mathrm{H}_n(K^R) = \mathrm{soc}(R)$ will also be important. Finally, for the case that $\dim(R) > 0$, consider the ideal J of R corresponding to $\mathrm{H}_1(K^R)^n \subseteq \mathrm{H}_n(K^R) = \mathrm{soc}(R)$ and apply Krull's Intersection Theorem to produce $c > 0$ where $J \not\subseteq \mathfrak{m}^c$. Consider the sequence $R \to R/\mathfrak{m}^{c+1} \to R/\mathfrak{m}^c$, and find a way to apply the depth zero case to show R is artinian giving a contradiction.