Macaulay2 Remark: It is OK to use `Macaulay2` for difficult calculations. For example, for computing kernels of differentials. Also, as a warning, the `acyclicClosure` command only produces the correct output when the ring contains $\mathbb{Q}$.

Exercise 1.* Let $R = \mathbb{F}_2[\![x,y]\!]/(x^2, xy)$. Let $R\langle Y\rangle$ denote the acyclic closure of k over R.

(a) Compute $R\langle Y_{\leqslant 4}\rangle$ without using the `Macaulay2` package `DGAlgebras`.

(b) Find the first 4 deviations of R. Use `Macaulay2` to verify your answer.

Exercise 2.* Let A be a graded-commutative dg R-algebra. Show that the natural map

$$A[x]\langle y \mid \partial y = x\rangle \longrightarrow A$$

is a homotopy equivalence of dg A-algebras.
Hint: First, a helpful reduction is to realize that there is an isomorphism of dg R-algebras

$$A \otimes_R R[x]\langle y \mid \partial y = x\rangle \cong A[x]\langle y \mid \partial y = x\rangle\,,$$

and so it suffices to show that $R[x]\langle y \mid \partial y = x\rangle \to R$ is a homotopy equivalence of dg R-algebras.

Exercise 3.$^{!!}$ Let x be a degree 2 variable and consider the graded algebras algebras $A = \mathbb{F}_2[x]$ and $B = \mathbb{F}_2\langle x\rangle$. Find the acyclic closures of $\mathbb{F}_2$ over and A and B, respectively.

Exercise 4. Let $R = Q/I$ with $(Q, \mathfrak{m}, k)$ a regular local ring and $I \subseteq \mathfrak{m}^2$ generated by a regular sequence. Use the quasiisomorphisms of local dg algebras (with residue field k)

$$K^R \xleftarrow{\simeq} Q[X_1] \otimes_Q K^Q \xrightarrow{\simeq} k[X_1]$$

and Problem Set 2 Exercise 6 to write down an explicit acyclic closure of k over R in terms of the following data[1]: $\mathfrak{m}$ is generated by $t_1, \dots, t_n$, the ideal I is minimally generated the regular sequence $f_1, \dots, f_c$ and

$$f_i = \sum_{j=1}^{n} a_{ij} t_j \quad \text{for some} \quad a_{ij} \in \mathfrak{m}\,.$$

Exercise 5. Does a similar explicit description for a minimal generating set for $\mathrm{H}_1(K^R)$ hold for a general R, as the one given in Exercise 4? If so, explain. If not, give a counterexample.

Exercise 6.* Fix a field k. Let A be a graded k-algebra. Check that A^{Lie} is a graded Lie algebra over k, and that the functor $A \to A^{\mathrm{Lie}}$ is right adjoint to $\mathrm{U}\colon \mathbf{Lie}_k \to \mathbf{Alg}_k$. Is this adjoint pair an equivalence? Justify your answer completely.

Exercise 7. Show that the center $\mathfrak{g}$ of a graded Lie algebra L is an abelian Lie subalgebra of L. Furthermore, this defines a central polynomial subalgebra $\mathrm{U}(\mathfrak{g})$ of $\mathrm{U}(L)$.

[1]This description is due to Tate.

Exercise 8.* Let k be a field and consider the graded k-vector space $L = k\{u_i, v_i \mid i \geqslant 0\}$ where $|u_i| = 2i$ and $|v_i| = 2i+1$, so that $\dim_k(L_i) = 1$ for all $i \geqslant 1$.

(a) Check that L is a graded Lie algebra under the antisymmetric bracket given by

$$[v_i, v_j] = 0 \qquad v_i^{[2]} = 0 \qquad [u_i, u_j] = 0 \qquad [u_i, v_j] = v_{i+j}.$$

(b) Find the central and nilpotent elements of L.

Exercise 9. For a graded Lie algebra L over k, show that $\mathrm{U}(L)$ is finite dimensional k-space if and only if L is finite dimensional and $L = L_{\mathrm{odd}}$.

Exercise 10. Show that $\mathrm{Lie}(V)$ is isomorphic to the Lie subalgebra of $\mathrm{T}(V)^{\mathrm{Lie}}$ generated by V.
Hint: Use the universal properties of the tensor algebra, the free Lie algebra, and the universal envelope to show that $\mathrm{U}(\mathrm{Lie}(V)) \cong \mathrm{T}(V)$. The Poincaré–Birkoff–Witt Theorem implies that $L \longrightarrow \mathrm{U}(L)^{\mathrm{Lie}}$ is injective for any graded Lie algebra L.

Exercise 11. Show that if $\mathrm{U}(L)$ is infinite dimensional, then $\mathrm{U}(L)$ is torsionfree. That is to say, prove that $\mathrm{tors}(\mathrm{U}(L)) = 0$ where

$$\mathrm{tors}(\mathrm{U}(L)) := \{y \in \mathrm{U}(L) \mid y x_{i_1} \cdots x_{i_n} = 0 \text{ for all } n \gg 0\}.$$

Exercise 12.* Let $Q = k[\![x, y]\!]$ and $R = Q/(x^2, xy)$, with a k a field of characteristic zero. Compute the brackets $[a, b]$ for $a, b \in \pi^i(R)$ with $i = 1, 2$. Check your work with `Macaulay2`.

Exercise 13.* Assume R is a complete intersection ring with regular presentation $\widehat{R} \cong Q/I$ satisfying that $I \subseteq \mathfrak{m}_Q^3$. Describe $\pi^*(R)$ as a graded Lie algebra over k, and use this description to calculate $\mathrm{Ext}_R(k, k)$ as a graded k-algebra.

Exercise 14.* For $R = k[\![x, y, z]\!]/(x^2 - yz, xz, y^2)$, use Problem Set 2 Exercise 4 to make a table for $\pi^*(R)$ recording the Lie brackets $[a, b]$ for $a, b \in \pi^i(R)$ with $i = 1, 2$.

Exercise 15. For

$$R = \frac{\mathbb{F}_7[\![x, y, z, w]\!]}{(xy, z^2, zw, x^2 z + w^3, y^2 w + xw^2)}$$

use `Macaulay2` to calculate the Lie brackets $[a, b]$ for $a, b \in \pi^i(R)$ with $i = 1, 2$.