Let A be a dg algebra over a fixed base (commutative) ring Q.

Exercise 1. Show that if M is a dg A-module, then $\mathrm{H}(M)$ is a graded $\mathrm{H}(A)$-module.

Exercise 2.* Decide if $R = k[\![x]\!]/(x^2)$ and

$$M = \quad \cdots \longrightarrow R \xrightarrow{x\cdot} R \xrightarrow{x\cdot} R \xrightarrow{x\cdot} R \longrightarrow \cdots$$

is a semiprojective dg R-module. Justify your answer completely.

Exercise 3. Let P be a dg A-module. Prove that P is semiprojective if and only if any/all of the following hold:

(a) For any $M \overset{\simeq}{\twoheadrightarrow} N$, the induced map $\mathrm{Hom}_A(P,M) \to \mathrm{Hom}_A(P,N)$ is a surjective quasiiso.

(b) P is a projective graded A-module and $\mathrm{Hom}_A(P,C) \simeq 0$ whenever $C \simeq 0$.

(c) P is a direct summand of a semifree dg A-module.

Exercise 4. What dg A-modules are the projective objects in the category of dg A-modules?

Exercise 5.* Prove that the mapping cone of semiprojective dg A-modules is also semiprojective.

Exercise 6. Prove the following facts about semifree dg modules.

(a) If F is semifree, then F is free as a graded A-module.

(b) If A is a nonnegatively graded Q-algebra (i.e., $A_i = 0$ for $i < 0$) and F is a dg A-module that is bounded below and F is free as a graded A-module, then F is semifree over A.

(c) If F is semifree, then F is semiprojective.

Exercise 7.* Let $Q = k[\![x]\!]$ and $f = x^3$. Set $E = Q[e \mid \partial e = f]$ and consider the Q-complex M defined by:

$$0 \to Q \xrightarrow{\begin{bmatrix} 0 \\ -x \end{bmatrix}} Q^2 \xrightarrow{\begin{bmatrix} x & 0 \end{bmatrix}} Q \to 0$$

concentrated in homological degrees 0,1,2.

(a) Check that the following maps prescribe M with distinct dg E-module structures:

$$0 \leftarrow Q \xleftarrow{\begin{bmatrix} 0 & -x^2 \end{bmatrix}} Q^2 \xleftarrow{\begin{bmatrix} x^2 \\ 0 \end{bmatrix}} Q \leftarrow 0 \quad \text{and} \quad 0 \leftarrow Q \xleftarrow{\begin{bmatrix} 1 & -x^2 \end{bmatrix}} Q^2 \xleftarrow{\begin{bmatrix} x^2 \\ 1 \end{bmatrix}} Q \leftarrow 0\,.$$

(b) Are the dg E-modules from part (a) the same up to homotopy equivalence (or, equivalently, quasiisomorphism)?

Exercise 8.* Let $Q = k[\![x,y]\!]$ and $f = x^3 + y^4$. Set $E = Q[e \mid \partial e = f]$ and $R = Q/(f)$. Consider the following complex F (concentrated in nonnegative degrees) that is periodic starting at homological degree 2.

$$\cdots \to Q^4 \xrightarrow{\begin{bmatrix} f & 0 & -y^3 & x^2 \\ 0 & f & x & y \\ -y & x^2 & 1 & 0 \\ x & y^3 & 0 & 1 \end{bmatrix}} Q^4 \xrightarrow{\begin{bmatrix} -1 & 0 & -y^3 & x^2 \\ 0 & -1 & x & y \\ -y & x^2 & -f & 0 \\ x & y^3 & 0 & -f \end{bmatrix}} Q^4 \xrightarrow{\begin{bmatrix} f & 0 & -y^3 & x^2 \\ 0 & f & x & y \\ -y & x^2 & 1 & 0 \\ x & y^3 & 0 & 1 \end{bmatrix}} Q^4 \xrightarrow{\begin{bmatrix} -y & x^2 & -f & 0 \\ x & y^3 & 0 & -f \\ 0 & -1 & x & y \end{bmatrix}} Q^3 \xrightarrow{\begin{bmatrix} x & y & f \end{bmatrix}} Q$$

(a) Prove that F is a free Q-resolution of k.[1]

(b) Prove that the matrices below give F a dg E-module structure

$$F_1 \xleftarrow{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}} F_0 \qquad F_2 \xleftarrow{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}} F_1$$

$$F_{2n+1} \xleftarrow{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}} F_{2n} \qquad F_{2n+2} \xleftarrow{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}} F_{2n+1}, \qquad \text{for } n \geqslant 1.$$

(c) Prove that F, equipped with the dg E-module structure from part (b), is a semifree resolution of k over E.

(d) Prove that $F \otimes_E R$ is the minimal free resolution of k over R.

[1] You may check this is correct with `Macaulay2`.