Exercise 1. (1.5) Let I be a nonzero ideal in a noetherian local ring R. Show that if I is generated by a regular sequence, then I/I^2 is a free R/I-module.

Exercise 2. (1.8) Prove that if $I \subseteq \mathfrak{m}^2$ is minimally generated by f, g in a regular local ring Q, then the minimal free resolution of Q/I can be equipped with a dg Q-algebra structure.

Exercise 3. (1.9) Let $I \subseteq \mathfrak{m}^2$ be minimally generated by f, g in a regular local ring $(Q, \mathfrak{m}, k)$, and let F denote the minimal free resolution of Q/I equipped with a dg Q-algebra structure from Exercise 2. Describe the induced graded k-algebra structure on $F \otimes_Q k$;.

Exercise 4. (1.10) For a pair of R-complexes M, N, their Hom complex, denoted $\operatorname{Hom}_R(M, N)$, is the R-complex with

$$\operatorname{Hom}_R(M,N)_i = \prod_{j\in\mathbb{Z}} \operatorname{Hom}_R(M_j, N_{i+j}) \quad \text{and} \quad \partial^{\operatorname{Hom}}(\varphi) = \partial^N \circ \varphi - (-1)^{|\varphi|}\varphi \circ \partial^M .$$

(a) Check that $\operatorname{Hom}_R(M, N)$ is a complex.

(b) Describe $\mathrm{Z}_0(\operatorname{Hom}_R(M, N))$ and $\mathrm{H}_0(\operatorname{Hom}_R(M, N))$.

(c) Set $\operatorname{End}_R(M) = \operatorname{Hom}_R(M, M)$. Prove that $\operatorname{End}_R(M)$ is a dg R-algebra.

Exercise 5. (2.9) Let $R = \mathbb{F}_5[\![x, y, z]\!]/(x^2, y^2, z^2, xyz)$.

(a) Argue numerically that R is not a complete intersection.

(b) Use `Macaulay2` to compute $\mathrm{H}(K^R)$, as a graded $\mathbb{F}_5$-algebra, and use this information to explain why R is not a complete intersection ring.

Exercise 6. (3.6) Fix a field k. Let A be a graded k-algebra. Check that A^{Lie} is a graded Lie algebra over k, and that the functor $A \to A^{\mathrm{Lie}}$ is right adjoint to $\mathrm{U}\colon \mathbf{Lie}_k \to \mathbf{Alg}_k$. Is this adjoint pair an equivalence? Justify your answer completely.

Exercise 7. (3.8) Let k be a field and consider the graded k-vector space $L = k\{u_i, v_i \mid i \geqslant 0\}$ where $|u_i| = 2i$ and $|v_i| = 2i + 1$, so that $\dim_k(L_i) = 1$ for all $i \geqslant 1$.

(a) Check that L is a graded Lie algebra under the antisymmetric bracket given by

$$[v_i, v_j] = 0 \qquad v_i^{[2]} = 0 \qquad [u_i, u_j] = 0 \qquad [u_i, v_j] = v_{i+j}.$$

(b) Find the central and nilpotent elements of L.

Exercise 8. (3.13) Assume R is a complete intersection ring with regular presentation $\widehat{R} \cong Q/I$ satisfying that $I \subseteq \mathfrak{m}_Q^3$. Describe $\pi^*(R)$ as a graded Lie algebra over k, and use this description to calculate $\operatorname{Ext}_R(k, k)$ as a graded k-algebra.

Exercise 9. (3.14) For $R = k[\![x, y, z]\!]/(x^2 - yz, xz, y^2)$, make a table for $\pi^*(R)$ recording the Lie brackets $[a, b]$ for $a, b \in \pi^i(R)$ with $i = 1, 2$.
You may use Macaulay2 to produce the minimal model of R over $k[\![x, y, z]\!]$.

Exercise 10. (4.6) Prove that the mapping cone of semiprojective dg A-modules is also semiprojective.

Exercise 11. (4.7) Let $Q = k[\![x]\!]$ and $f = x^3$. Set $E = Q[e \mid \partial e = f]$ and consider the Q-complex M defined by:

$$0 \to Q \xrightarrow{\begin{bmatrix} 0 \\ -x \end{bmatrix}} Q^2 \xrightarrow{\begin{bmatrix} x & 0 \end{bmatrix}} Q \to 0$$

concentrated in homological degrees 0,1,2.

(a) Check that the following maps prescribe M with distinct dg E-module structures:

$$0 \leftarrow Q \xleftarrow{\begin{bmatrix} 0 & -x^2 \end{bmatrix}} Q^2 \xleftarrow{\begin{bmatrix} x^2 \\ 0 \end{bmatrix}} Q \leftarrow 0 \quad \text{and} \quad 0 \leftarrow Q \xleftarrow{\begin{bmatrix} 1 & -x^2 \end{bmatrix}} Q^2 \xleftarrow{\begin{bmatrix} x^2 \\ 1 \end{bmatrix}} Q \leftarrow 0\,.$$

(b) Are the dg E-modules from part (a) homotopy equivalent dg E-modules?

Hint: If M and N are homotopy equivalent dg E-modules, then $M \otimes_R k$ and $N \otimes_Q k$ are homotopy equivalent dg $E \otimes_Q k$-modules.

Exercise 12. (4.8) Let $Q = k[\![x, y]\!]$ and $f = x^3 + y^4$. Set $E = Q[e \mid \partial e = f]$ and $R = Q/(f)$. Consider the following complex F (concentrated in nonnegative degrees) that is periodic starting at homological degree 2.

$$\cdots \to Q^4 \xrightarrow{\begin{bmatrix} f & 0 & -y^3 & x^2 \\ 0 & f & x & y \\ -y & x^2 & 1 & 0 \\ x & y^3 & 0 & 1 \end{bmatrix}} Q^4 \xrightarrow{\begin{bmatrix} -1 & 0 & -y^3 & x^2 \\ 0 & -1 & x & y \\ -y & x^2 & -f & 0 \\ x & y^3 & 0 & -f \end{bmatrix}} Q^4 \xrightarrow{\begin{bmatrix} f & 0 & -y^3 & x^2 \\ 0 & f & x & y \\ -y & x^2 & 1 & 0 \\ x & y^3 & 0 & 1 \end{bmatrix}} Q^4 \xrightarrow{\begin{bmatrix} -y & x^2 & -f & 0 \\ x & y^3 & 0 & -f \\ 0 & -1 & x & y \end{bmatrix}} Q^3 \xrightarrow{\begin{bmatrix} x & y & f \end{bmatrix}} Q$$

(a) Prove that F is a free Q-resolution of k.[1]

(b) Prove that the matrices below give F a dg E-module structure

$$F_1 \xleftarrow{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}} F_0 \qquad F_2 \xleftarrow{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}} F_1$$

$$F_{2n+1} \xleftarrow{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}} F_{2n} \qquad F_{2n+2} \xleftarrow{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}} F_{2n+1}, \qquad \text{for } n \geqslant 1.$$

(c) Prove that F, equipped with the dg E-module structure from part (b), is a semifree resolution of k over E.

(d) Prove that $F \otimes_E R$ is the minimal free resolution of k over R.

[1]You may check this is correct with `Macaulay2`.